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Working with multiscale asymptotics

Solving weakly nonlinear oscillator equations on long-time intervals

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Abstract. This paper surveys, compares and updates techniques to obtain the asymptotic solution of the weakly
nonlinear oscillator equation $\ddot{y} + y + \epsilon f(y, \dot{y}) = 0$ as $\epsilon \to 0$ and for corresponding first-order vector systems. The
solutions found by the regular perturbation method generally feature resonance and so break down as $t \to \infty$.
The classical methods of averaging and multiple scales eliminate such secular behavior and provide asymptotic
solutions valid for time intervals of length $t = \mathcal{O}(\epsilon^{-1})$. The renormalization group method proposed by Chen
et al. [*Phys. Rev. E* 54 (1996) 376–394] gives equivalent results. Several well-known examples are solved with these
methods to demonstrate the respective techniques and the equivalency of the approximations produced. Finally,
an amplitude-equation method is derived that incorporates the best features of all these techniques. This method
is both straightforward to automate with a computer-algebra system and flexible enough to allow the forcing f
to depend on the small parameter.

Key words: amplitude equation, averaging, multiple scales, oscillations, renormalization, singular perturbations

> *This paper is dedicated to Jerry Kevorkian with sincere appreciation for his long career
> of dedicated teaching at the University of Washington and for his substantial
> contributions to multiscale asymptotics.*

1. Introduction

Mathematical models displaying nonlinear oscillations are ubiquitous in the vast scientific lit-
erature. Through the years, a number of specialized techniques have evolved to analyze these
oscillatory problems. Having arisen from differing needs, these techniques all appear to be
(and are treated as) fundamentally different in character. For the class of problems discussed
in this paper, however, we will show that these differences are largely superficial.

The purpose of this paper is threefold. First, we will (re)acquaint readers with the popu-
lar methods for asymptotically solving weakly nonlinear oscillator equations. Second, we will
demonstrate the equivalency of these different methods. Last, we will synthesize the best fea-
tures of the techniques presented to produce an efficient amplitude-equation approach. We
begin our discussion with the regular perturbation expansion: the most obvious, but also most
limited, path to an asymptotic solution.

Asymptotically solving the autonomous, weakly nonlinear oscillator equation

$$\ddot{y} + y + \epsilon f(y, \dot{y}) = 0, \tag{1.1}$$

for f smooth and ϵ small and positive, is a classical problem of nonlinear oscillations [1, Part 2]. Using the initial conditions

$$y(0) = 1 \quad \text{and} \quad \dot{y}(0) = 0, \tag{1.2}$$

one naturally seeks an approximate solution as a truncation of the unique formal power series

$$y_\epsilon(t) = \sum_{j=0}^{\infty} \epsilon^j y_j(t), \tag{1.3}$$

that one obtains by expanding in ϵ about the solution

$$y_0(t) = \cos t$$

of the reduced ($\epsilon = 0$) problem. Regular perturbation theory shows how to find successive terms in the series by equating coefficients of like powers of ϵ in the differential equation and initial conditions [2]. The expansion for y directly implies those for $\dot{y}$ and $\ddot{y}$, while the Maclaurin expansion for $f(y_\epsilon, \dot{y}_\epsilon)$ is

$$f(y_\epsilon, \dot{y}_\epsilon) = f(y_0, \dot{y}_0) + \epsilon \left[f_y(y_0, \dot{y}_0) y_1 + f_{\dot{y}}(y_0, \dot{y}_0) \dot{y}_1 \right] + \mathcal{O}(\epsilon^2).$$

From the coefficients of ϵ^j for $j = 0, 1$, and 2, we thereby obtain a sequence of linear initial-value problems

$$\ddot{y}_0 + y_0 = 0, \quad y_0(0) = 1, \quad \dot{y}_0(0) = 0,$$
$$\ddot{y}_1 + y_1 + f(y_0, \dot{y}_0) = 0, \quad y_1(0) = 0, \quad \dot{y}_1(0) = 0$$

and

$$\ddot{y}_2 + y_2 + f_y(y_0, \dot{y}_0) y_1 + f_{\dot{y}}(y_0, \dot{y}_0) \dot{y}_1 = 0, \quad y_2(0) = 0, \quad \dot{y}_2(0) = 0.$$

More generally, the coefficient y_j for each $j \geq 1$ will satisfy an initial-value problem

$$\ddot{y}_j + y_j = g_{j-1}(y_0, \dot{y}_0, \ldots, y_{j-1}, \dot{y}_{j-1}),$$

with trivial initial conditions at $t = 0$ and with the forcing g_{j-1} being known successively. Variation of parameters immediately implies that

$$y_j(t) = \int_0^t \sin(t - s) g_{j-1}(y_0(s), \ldots, \dot{y}_{j-1}(s)) \mathrm{d}s, \quad j \geq 1. \tag{1.4}$$

Thus,

$$y_1(t) = -\int_0^t \sin(t - s) f(\cos s, -\sin s) \mathrm{d}s,$$

and

$$y_2(t) = \int_0^t \sin(t - s) \left[f_y(\cos s, -\sin s) \int_0^s \sin(s - r) f(\cos r, -\sin r) \mathrm{d}r \right.$$
$$\left. + f_{\dot{y}}(\cos s, -\sin s) \int_0^s \cos(s - r) f(\cos r, -\sin r) \mathrm{d}r \right] \mathrm{d}s.$$

Using Gronwall-inequality estimates, one may easily show that the series (1.3) obtained (as well as those for the derivatives) converges on any *finite* t interval for ϵ sufficiently small [3, Section 4.3], [4, Sections 3.2, 3.3]. Without significant change, the stated conclusions also

apply to nonautonomous equations, $\ddot{y} + y + \epsilon f(y, \dot{y}, t, \epsilon)$, [5, Chapters 2–3]. To illustrate the complications that can arise when t becomes unbounded, let us consider some examples.

We first examine the nonautonomous, linear equation

$$\ddot{y} + y = \epsilon \sin t. \tag{1.5}$$

Given the initial values (1.2), the exact solution is

$$y(t) = \cos t + \frac{\epsilon}{2}(\sin t - t \cos t). \tag{1.6}$$

Considered as a two-term regular perturbation series, (1.6) is correct but only asymptotically valid for finite t because of the secular term $t \cos t$ in the $\mathcal{O}(\epsilon)$ coefficient. For times $t \gg \mathcal{O}(\epsilon^{-1})$, the magnitude of the second term in the series exceeds that of the first. This complication results because the forcing, $\epsilon \sin t$, lies in the nullspace of the unperturbed harmonic operator.

A singular perturbation problem arises when the regular perturbation method is no longer uniformly valid. The technique might break down, for example, either for large t-values (as for (1.6)) or in the presence of boundary or interior layers [6, Chapter 1], [7, Chapter 10]. Such problems are generally solved using *asymptotic series* [8, Chapter 8], [9, Chapter 12], [10], for which one typically uses only the first few terms. We naturally insist that the accuracy of the successive approximations defined by the truncations improve in the $\epsilon \to 0$ limit. For (1.6), the asymptotic ordering no longer holds when t becomes unbounded, since the second term ultimately dominates the first. One calls the second term *secular* as $t \to \infty$, because it increases without bound, thus violating the implicit assumption that successive terms in an asymptotic expansion remain of decreasing size. The appearance of secular terms heralds the unsuitability of the regular perturbation expansion when t becomes large.

A second prototype example, the Duffing equation,

$$\ddot{y} + y + \epsilon y^3 = 0, \tag{1.7}$$

describes the motion of a slightly nonlinear spring. Using the initial conditions (1.2), the two-term regular perturbation solution is

$$y_\epsilon(t) = \cos t + \epsilon \left(-\frac{3}{8} t \sin t - \frac{1}{32} \cos t + \frac{1}{32} \cos 3t \right) + \mathcal{O}\left(\epsilon^2 t^2 \right). \tag{1.8}$$

This unbounded approximation is unacceptable as $t \to \infty$ because a bounded, periodic solution is given in terms of elliptic integrals. Specifically, one can integrate (1.7) once to get a conserved energy and then separate variables to get the implicit solution

$$t = \mp \int_y^1 \frac{\mathrm{d}r}{\sqrt{(1 - r^2)(1 + \frac{\epsilon}{2}(1 + r^2))}}, \tag{1.9}$$

depending smoothly on ϵ. An elementary phase-plane analysis shows that the solution has the ϵ-dependent period

$$T(\epsilon) = 2 \int_{-1}^1 \frac{\mathrm{d}r}{\sqrt{(1 - r^2)(1 + \frac{\epsilon}{2}(1 + r^2))}} = 2\pi \left(1 - \frac{3}{8}\epsilon + \frac{57}{256}\epsilon^2 + \mathcal{O}(\epsilon^3) \right).$$

This suggests that the regular perturbation approximation incorrectly accounts for the natural frequency, tacitly assuming it to be one, rather than

$$\frac{2\pi}{T(\epsilon)} = 1 + \frac{3}{8}\epsilon - \frac{21}{256}\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{1.10}$$

The regular perturbation method breaks down when $t \to \infty$, due to the appearance of *spurious* secular terms. We call them spurious because they are an artifact of the perturbation procedure, not inherent in the problem. One must therefore find a way to *exorcize* these spurious terms from the regular perturbation series if it is to retain its value as an asymptotic expansion for large t. This complication has been a long-recognized phenomenon in celestial mechanics [11, Vol. I, Part 2], [12, Chapters 6–10]. Lindstedt, [13], cleverly introduced the natural *strained coordinate*

$$\tilde{t} = \frac{2\pi}{T(\epsilon)} t \equiv (1 + \epsilon\Omega(\epsilon))t, \tag{1.11}$$

using an unspecified asymptotic series

$$\Omega(\epsilon) \sim \Omega_0 + \epsilon\Omega_1 + \epsilon^2\Omega_2 + \cdots \tag{1.12}$$

for the scaled frequency. The individual terms of (1.12) are determined successively, together with a *generalized* asymptotic expansion [14], [15, pp. 24–27],

$$y(\tilde{t}, \epsilon) \sim y_0(\tilde{t}) + \epsilon y_1(\tilde{t}) + \cdots \tag{1.13}$$

for the solution, when one insists that successive y_k's be periodic functions of $\tilde{t}$. This condition implies selection criteria for each coefficient Ω_{k-1}: to eliminate *resonant* forcing in the resulting differential equation for the corresponding y_k. In terms of $\tilde{t}$, y satisfies the transformed initial-value problem

$$(1 + \epsilon\Omega(\epsilon))^2 \frac{d^2 y}{d\tilde{t}^2} + y + \epsilon y^3 = 0, \quad y(0) = 1, \quad \frac{dy}{d\tilde{t}}(0) = 0. \tag{1.14}$$

Applying the regular perturbation process to (1.14) leads to a sequence of linear problems, beginning with

$$\frac{d^2 y_0}{d\tilde{t}^2} + y_0 = 0, \quad y_0(0) = 1, \quad \frac{dy_0}{d\tilde{t}}(0) = 0$$

and

$$\frac{d^2 y_1}{d\tilde{t}^2} + y_1 + 2\Omega_0 \frac{d^2 y_0}{d\tilde{t}^2} + y_0^3 = 0, \quad y_1(0) = \frac{dy_1}{d\tilde{t}}(0) = 0.$$

Since $y_0(\tilde{t}) = \cos\tilde{t}$, y_1 must be a periodic solution of

$$\frac{d^2 y_1}{d\tilde{t}^2} + y_1 = 2\Omega_0 y_0 - y_0^3 \equiv \left(2\Omega_0 - \frac{3}{4}\right)\cos\tilde{t} - \frac{1}{4}\cos 3\tilde{t}.$$

The complimentary solutions are linear combinations of $\cos\tilde{t}$ and $\sin\tilde{t}$, so resonant solutions arise if and only if the forcing term includes multiples of either of them. This solvability, or Fredholm alternative, consideration specifies the resonant terms that we need to eliminate in (1.13).

To avoid secular behavior in y_1, we must pick $\Omega_0 = 3/8$ to uniquely obtain

$$y_1(\tilde{t}) = -\frac{1}{32}\left(\cos\tilde{t} - \cos 3\tilde{t}\right).$$

The process of successively defining the terms of the series for y and for Ω can be continued to any order. At the next stage, one gets $\Omega_1 = -\frac{21}{256}$ and

$$y_2(\tilde{t}) = \frac{1}{1024}\left(23\cos\tilde{t} - 24\cos 3\tilde{t} + \cos 5\tilde{t}\right).$$

Note that the selections for Ω_0 and Ω_1 agree with the ϵ and ϵ^2 coefficients in the power series (1.10) for the frequency. Using the truncated series suggests that one could interpret the resulting approximation as a multi-time expansion for a finite number of times $t_k \equiv \epsilon^k t$, $k = 0, 1, 2, \ldots$, [4, Section 4.8].

A third example is the slightly damped linear oscillator described by

$$\ddot{y} + y + 2\epsilon \dot{y} = 0. \tag{1.15}$$

The unique solution of the initial-value problem (1.15), (1.2) is

$$y_\epsilon(t) = e^{-\epsilon t} \left(\cos(\sqrt{1 - \epsilon^2} t) + \frac{\epsilon}{\sqrt{1 - \epsilon^2}} \sin(\sqrt{1 - \epsilon^2} t) \right).$$

It is critical to note that it depends on both the *slow* time $\sigma = \epsilon t$, which describes its ultimate decay, and the *fast* time $\eta = \sqrt{1 - \epsilon^2} t$, which scales its damped oscillations. Neither the regular perturbation approach nor its modification as the method of strained coordinates will suffice to provide the asymptotic solution on long time intervals, though the regular expansion procedure is adequate for t finite and ϵ sufficiently small.

In the remainder of this paper we present various ways to improve upon the regular perturbation results. In Section 2, we discuss the method of *multiple scales*. To prepare for later sections, we detour in Section 3 to reformulate (1.1) as a vector system. The classical method of *averaging* is considered in Section 4. In Section 5, we explore the *renormalization group* method, recently popular in the physics literature [16]. Lastly, we present our own synthesis and extension of these disparate methods in what we refer to as the *amplitude equation* technique (see also [17], where the method is discussed in greater detail and applied to a wider variety of examples). The amplitude-equation technique is quite easily automated with a computer-algebra system. Additionally, one may use the amplitude equations to find ϵ-dependent frequencies for such weakly nonlinear oscillators, since their rest points provide the radius of any limit cycle and its frequency.

2. Multiple scales

The method of *multiple scales*, popularly known as *two-timing*, was primarily developed post-Sputnik at Caltech by Kevorkian and Cole [18, Chapter 4], [19]. Kuzmak [20] developed similar ideas for certain strictly nonlinear oscillators. Multiple Scales has been successfully and extensively applied throughout the sciences for over forty years. We shall illustrate the method by applying it to the oscillator problem (1.1).

We seek an asymptotic solution using the formal two-time expansion

$$y_\epsilon(\eta, \sigma) \sim y_0(\eta, \sigma) + \epsilon y_1(\eta, \sigma) + \epsilon^2 y_2(\eta, \sigma) + \cdots \tag{2.1}$$

for a *fast* time

$$\eta \equiv (1 + \epsilon^2 \omega(\epsilon)) t,$$

with some power series

$$\omega(\epsilon) = \omega_0 + \omega_1 \epsilon + \cdots,$$

and the *slow* time

$$\sigma = \epsilon t.$$

The series for y and ω will be obtained termwise in a coordinated manner, treating η and σ as independent variables. Strained coordinates is thereby a special case of multiple scales that corresponds to using the single linearly combined time $\eta + \tilde{\Omega}(0)\sigma \equiv (1 + \epsilon\tilde{\Omega}(\epsilon))t$.

Applying the chain rule, we find that the second derivative is

$$\ddot{y} = \left(1 + \epsilon^2\omega(\epsilon)\right)^2 \frac{\partial^2 y}{\partial\eta^2} + 2\epsilon \left(1 + \epsilon^2\omega(\epsilon)\right) \frac{\partial^2 y}{\partial\eta\partial\sigma} + \epsilon^2 \frac{\partial^2 y}{\partial\sigma^2},$$

so a Maclaurin expansion in ϵ imples the first three terms of (2.1) must satisfy

$$\frac{\partial^2 y_0}{\partial\eta^2} + y_0 = 0, \tag{2.2}$$

$$\frac{\partial^2 y_1}{\partial\eta^2} + y_1 = -\left[2\frac{\partial^2 y_0}{\partial\eta\partial\sigma} + f\left(y_0, \frac{\partial y_0}{\partial\eta}\right)\right] \tag{2.3}$$

and

$$\frac{\partial^2 y_2}{\partial\eta^2} + y_2 = -\left[2\frac{\partial^2 y_1}{\partial\eta\partial\sigma} + 2\omega_0\frac{\partial^2 y_0}{\partial\eta^2} + \frac{\partial^2 y_0}{\partial\sigma^2} + \frac{\partial f}{\partial y}\left(y_0, \frac{\partial y_0}{\partial\eta}\right) y_1 \right.$$
$$\left. + \frac{\partial f}{\partial\dot{y}}\left(y_0, \frac{\partial y_0}{\partial\eta}\right)\left(\frac{\partial y_1}{\partial\eta} + \frac{\partial y_0}{\partial\sigma}\right)\right]. \tag{2.4}$$

This sequence of equations may be continued to any order in ϵ. We collapse the notation by writing

$$\mathcal{L}[y_k] = g_{k-1}(\eta, \sigma), \quad k \geq 1, \tag{2.5}$$

for the operator $\mathcal{L} \equiv \frac{\partial^2}{\partial\eta^2} + 1$ and forcings g_{k-1} known successively in terms of y_0, y_1, ..., y_{k-1}. We will solve Equations (2.5) subject to initial conditions that follow from expanding $y(0) = y_\epsilon(0, 0) = 1$ and

$$\dot{y}(0) = \left(1 + \epsilon^2\omega(\epsilon)\right)\frac{\partial y_\epsilon}{\partial\eta}(0, 0) + \epsilon\frac{\partial y_\epsilon}{\partial\sigma}(0, 0) = 0$$

as power series in ϵ. The bounded coefficient y_k will be specified by eliminating resonance in the forcing g_{k-1}.

Using polar coordinates for convenience, we observe that the homogeneous, leading-order equation (2.2) has the solution

$$y_0(\eta, \sigma) = A_0(\sigma)\cos(\eta + \phi_0(\sigma)), \tag{2.6}$$

with the amplitude, A_0, and the phase shift, ϕ_0, as "constants" of the integration with respect to η. They remain free functions of the slow time σ, except for the initial values $A_0(0) = 1$ and $\phi_0(0) = 0$. At this stage, we introduce the shifted phase

$$\Psi \equiv \eta + \phi_0(\sigma),$$

so that $\frac{\partial}{\partial\eta} = \frac{\partial}{\partial\Psi}$ and $y_0 = A_0 \cos\Psi$. For $k = 1$, (2.5) can then be rewritten as

$$\mathcal{L}[y_1] = g_0(\Psi, \sigma) \equiv 2A_0\frac{d\phi_0}{d\sigma}\cos\Psi + 2\frac{dA_0}{d\sigma}\sin\Psi - f_0(A_0, \Psi),$$

where $f_0(A_0, \Psi) \equiv f(A_0\cos\Psi, -A_0\sin\Psi)$. Its solution takes the form

$$y_1(\Psi, \sigma) = A_1(\sigma)\cos\Psi + B_1(\sigma)\sin\Psi + y_1^{(p)}(\Psi, \sigma), \tag{2.7}$$

for any particular solution $y_1^{(p)}(\Psi, \sigma)$. Since g_0 is a 2π-periodic function of Ψ, it can be expanded in a Fourier series. Thus, one can invoke the Fredholm alternative to show that secular terms occur in y_1 unless g_0 is orthogonal to both linearly independent solutions, $\cos\Psi$ and $\sin\Psi$, of $\mathcal{L}[y] = 0$. We therefore require that first-order harmonics be absent in the Fourier expansion for $g_0(\Psi, \sigma)$. The Fourier series for $f_0(A_0, \Psi)$ is

$$f_0(A_0, \Psi) = \alpha_0(A_0) + 2\sum_{n=1}^{\infty} (\alpha_n(A_0)\cos n\Psi + \beta_n(A_0)\sin n\Psi),$$

where the cosine and sine coefficients are

$$\alpha_n(A_0) = \frac{1}{2\pi} \int_0^{2\pi} f_0(A_0, \Psi)\cos n\Psi \, \mathrm{d}\Psi \equiv \langle f_0(A_0, \Psi)\cos n\Psi \rangle \tag{2.8}$$

and

$$\beta_n(A_0) = \frac{1}{2\pi} \int_0^{2\pi} f_0(A_0, \Psi)\sin n\Psi \, \mathrm{d}\Psi \equiv \langle f_0(A_0, \Psi)\sin n\Psi \rangle. \tag{2.9}$$

Here we use the traditional notation

$$\langle g(\Psi, \sigma) \rangle = \frac{1}{2\pi} \int_0^{2\pi} g(\Psi, \sigma) \, \mathrm{d}\Psi \tag{2.10}$$

for the constant average of a 2π-periodic function of Ψ. To eliminate resonance, we select the unspecified multiples $2\frac{\mathrm{d}A_0}{\mathrm{d}\sigma}$ and $2A_0\frac{\mathrm{d}\phi_0}{\mathrm{d}\sigma}$, of $\sin\Psi$ and $\cos\Psi$, respectively, in g_0, to satisfy the nonlinear triangular system

$$A_0 \frac{\mathrm{d}\phi_0}{\mathrm{d}\sigma} = \alpha_1(A_0), \tag{2.11}$$

$$\frac{\mathrm{d}A_0}{\mathrm{d}\sigma} = \beta_1(A_0), \tag{2.12}$$

which follows from the orthogonality conditions

$$\langle g_0(\Psi, \sigma)\cos\Psi \rangle = \langle g_0(\Psi, \sigma)\sin\Psi \rangle = 0.$$

The system (2.11), (2.12) will be solved subject to $\phi_0(0) = 0$ and $A_0(0) = 1$.

The initial-value problem for the planar system (2.11), (2.12) is uniquely solvable near $\sigma = 0$. In the corresponding phase plane, it reduces to integrating

$$\frac{\mathrm{d}\phi_0}{\mathrm{d}A_0} = \frac{\alpha_1(A_0)}{A_0\beta_1(A_0)}$$

along the trajectory uniquely determined by the initial values. If $\beta_1(A_0) \equiv 0$, we have $A_0(\sigma) \equiv 1$ and $\phi_0(\sigma) = \alpha_1(1)\sigma$. Otherwise, we take A_0 as the independent variable to locally provide the implicit solution

$$\sigma = \int_1^{A_0} \frac{\mathrm{d}r}{\beta_1(r)}, \tag{2.13}$$

$$\phi_0(\sigma) = \int_1^{A_0} \frac{\alpha_1(r)}{r\beta_1(r)} \mathrm{d}r. \tag{2.14}$$

As long as $A_0(\sigma)$ remains defined, the limiting asymptotic solution $y_0(\Psi, \sigma)$ is given by (2.6). If the solution to (2.12) continues to exist as $\sigma \to \infty$, the rest point, $A_0(\infty)$, attained must be the nearest zero of β_1 below 1 (or, respectively, above 1) if $\beta_1(1) < 0$ (or $\beta_1(1) > 0$). If this is an asymptotically stable rest point, the exponential decay will provide an appropriate limiting approximation, y_0, for all $t \geq 0$. We always determine y_0 on some $t = \mathcal{O}(\epsilon^{-1})$ interval, beyond the $t = \mathcal{O}(1)$ interval where regular perturbation theory is justified. It no longer suffices to solve successive linear systems, however; the nonlinear system (2.11), (2.12) now becomes basic.

Knowing A_0 and ϕ_0, (2.5) for $k = 1$ becomes

$$\mathcal{L}[y_1] = \tilde{g}_0(\Psi, \sigma) \equiv 2\alpha_1(A_0) \cos \Psi + 2\beta_1(A_0) \sin \Psi - f_0(A_0, \Psi),$$

where $\tilde{g}_0$ is a nonresonant forcing. We now seek a particular solution,

$$y_1^{(p)}(\Psi, \sigma) = u(A_0, \Psi) \cos \Psi + A_0 \, v(A_0, \Psi) \sin \Psi, \tag{2.15}$$

where variation of parameters implies that the coefficients u and v must satisfy

$$\frac{\partial u}{\partial \Psi} = f_0(A_0, \Psi) \sin \Psi - \beta_1(A_0), \tag{2.16}$$

$$-A_0 \frac{\partial v}{\partial \Psi} = f_0(A_0, \Psi) \cos \Psi - \alpha_1(A_0). \tag{2.17}$$

We shall integrate these ODEs subject to the auxiliary conditions

$$\int_0^{2\pi} u(A_0, \Psi) \, \mathrm{d}\Psi = \int_0^{2\pi} v(A_0, \Psi) \, \mathrm{d}\Psi = 0, \tag{2.18}$$

so that u and v will each have a *zero average*. Thus,

$$y_1(\Psi, \sigma) = [A_1(\sigma) + u(A_0, \Psi)] \cos \Psi + [B_1(\sigma) + A_0 \, v(A_0, \Psi)] \sin \Psi. \tag{2.19}$$

The particular solution (2.15) may contain first harmonic terms that one could absorb into the constants $A_1(\sigma)$ and $B_1(\sigma)$. We choose this particular solution, however, since it simplifies the derivation of the evolution equations for A_1 and B_1. Making this choice, of course, also affects the initial values $A_1(0)$ and $B_1(0)$.

To complete the $\mathcal{O}(\epsilon)$ approximation, y_1, we specify the evolution of $A_1(\sigma)$ and $B_1(\sigma)$ by eliminating resonant terms (*i.e.*, first harmonics) from (2.5) for $k = 2$. We therefore impose the nonsecularity (orthogonality) conditions

$$\langle g_1(\Psi, \sigma) \cos \Psi \rangle = \langle g_1(\Psi, \sigma) \sin \Psi \rangle = 0.$$

This provides the linear, triangular system

$$\frac{\mathrm{d}A_1}{\mathrm{d}\sigma} - \frac{\mathrm{d}\beta_1}{\mathrm{d}A_0} A_1 = \mathcal{B}(A_0), \tag{2.20}$$

$$\frac{\mathrm{d}B_1}{\mathrm{d}\sigma} - \frac{\beta_1}{A_0} B_1 = \left(\frac{\alpha_1}{A_0} - \frac{\mathrm{d}\alpha_1}{\mathrm{d}A_0} \right) A_1 + \omega_0 A_0 - \mathcal{A}(A_0), \tag{2.21}$$

where

$$(\mathcal{A}(A_0), \mathcal{B}(A_0)) = \frac{1}{2\pi} \int_0^{2\pi} \left(u(A_0, \Psi) \frac{\partial f_0}{\partial A_0} - v(A_0, \Psi) \frac{\partial f_0}{\partial \Psi} \right) (\cos \Psi, \sin \Psi) \, \mathrm{d}\Psi.$$

For a detailed derivation of (2.20) and (2.21), we refer the interested reader to [18, Section 4.2.5], [21] or [22]. The needed initial conditions $A_1(0)$ and $B_1(0)$ are found from $y_1(0, 0) = 0$ and $\frac{\partial y_1}{\partial \Psi}(0, 0) = -\frac{\partial y_0}{\partial \sigma}(0, 0)$.

Since $d\sigma = \frac{dA_0}{\beta_1(A_0)}$, an integration determines A_1 as a function of A_0, *viz.*

$$A_1(A_0) = \beta_1(A_0) \left[\frac{A_1(1)}{\beta_1(1)} + \int_1^{A_0} \frac{\mathcal{B}(r)}{\beta_1^2(r)} \, dr \right] \tag{2.22}$$

(unless $\beta_1(A_0) \equiv 0$, when $A_1(\sigma) = A_1(0) + \mathcal{B}(1)\sigma$). Knowing A_1,

$$B_1(A_0) = A_0 \left[B_1(1) + \int_1^{A_0} \frac{1}{\beta_1(r)} \left[\omega_0 - A_1(r) \frac{d}{dr}\left(\frac{\alpha_1(r)}{r}\right) - \frac{\mathcal{A}(r)}{r} \right] dr \right]. \tag{2.23}$$

This determines $y_1(\Psi, \sigma)$ as long as A_0 is defined. However, it involves resonance as $\sigma \to \infty$, where β_1 vanishes, unless we select

$$\omega_0 = \left(A_1(A_0) \frac{d}{dA_0}\left(\frac{\alpha_1(A_0)}{A_0}\right) + \frac{\mathcal{A}(A_0)}{A_0} \right) \Bigg|_{A_0(\infty)}. \tag{2.24}$$

Higher-order terms in the series for y and ω follow analogously, without complication. We recover the regular perturbation result (1.3) when we re-expand the multiple scale expansion (2.1) asymptotically for finite t's. Thus, multiple scales extends the regular perturbation result, which is based on a naive expansion, to a longer time interval by appropriately using the solution of the nonlinear system (2.11), (2.12) for the limiting amplitude and phase shift. We point out that [23] provides a proof of two-timing's asymptotic correctness for bounded σ. To fix ideas, we now demonstrate the procedure.

As a first example, recall Duffing's equation, (1.7). The forcing is $f(y, \dot{y}) = y^3$, so

$$f_0(A_0, \Psi) = \frac{3}{4} A_0^3 \cos \Psi + \frac{1}{4} A_0^3 \cos 3\Psi.$$

By inspection, the first-harmonic Fourier coefficients are $\alpha_1(A_0) = \frac{3}{8} A_0^3$ and $\beta_1(A_0) = 0$, so A_0 and ϕ_0 must satisfy

$$\frac{dA_0}{d\sigma} = 0, \quad \text{and} \quad \frac{d\phi_0}{d\sigma} = \frac{3}{8} A_0^2.$$

Since $A_0(0) = 1$ and $\phi_0(0) = 0$, we find the unique limiting solution

$$y_0(\eta, \sigma) = \cos\left(\eta + \frac{3}{8}\sigma\right),$$

in agreement with the Poincaré-Lindstedt method. At $\mathcal{O}(\epsilon)$, we obtain

$$y_1(\Psi, \sigma) = A_1(\sigma) \cos \Psi + B_1(\sigma) \sin \Psi + \frac{1}{32} (\cos 3\Psi - \cos \Psi),$$

where A_1 and B_1 must satisfy the decoupled problems

$$\frac{dA_1}{d\sigma} = 0, \quad A_1(0) = \frac{5}{32},$$

and

$$\frac{dB_1}{d\sigma} = \omega_0 + \frac{21}{256}, \quad B_1(0) = 0.$$

$B_1(\sigma)$ will be bounded as $\sigma \to \infty$ if and only if we choose

$$\omega_0 = -\frac{21}{256}.$$

Thus the asymptotic solution to $\mathcal{O}(\epsilon)$ is

$$y(t) = \cos \Psi + \frac{\epsilon}{32} (\cos 3\Psi - \cos \Psi) + \mathcal{O}(\epsilon^2),$$

where

$$\Psi = \left(1 + \frac{3}{8}\epsilon - \frac{21}{256}\epsilon^2 + \mathcal{O}(\epsilon^3)\right) t.$$

This approximation is valid for $t = \mathcal{O}(\epsilon^{-1})$.

As a second example, recall the linear oscillator (1.15). The forcing is $f(y, \dot{y}) = 2\dot{y}$, so

$$f_0(A_0, \Psi) = -2A_0 \sin \Psi,$$

has the first Fourier coefficients $\alpha_1(A_0) = 0$ and $\beta_1(A_0) = -A_0$. Thus,

$$\frac{dA_0}{d\sigma} = -A_0 \quad \text{and} \quad \frac{d\Phi}{d\sigma} = 0.$$

Because $A_0(0) = 1$ and $\phi_0(0) = 0$,

$$A_0(\sigma) = e^{-\sigma} \quad \text{and} \quad \phi_0(\sigma) = 0.$$

Proceeding further, A_1 and B_1 must satisfy

$$\frac{d}{dA_0}\left(\frac{A_1(A_0)}{A_0}\right) = 0, \quad A_1(1) = 0,$$

and

$$\frac{d}{dA_0}\left(\frac{B_1(A_0)}{A_0}\right) = -\frac{1}{A_0}\left(\omega_0 + \frac{1}{2}\right), \quad B_1(1) = \frac{1}{2}.$$

Thus $A_1(A_0) \equiv 0$, and $B_1(A_0)$ will be secular as $\sigma \to \infty$ unless

$$\omega_0 = -\frac{1}{2}.$$

Then, $B_1(A_0) = \frac{1}{2}A_0$, so the approximate solution is

$$y(t) = e^{-\sigma} (\cos \Psi + \epsilon \sin \Psi) + \mathcal{O}(\epsilon^2),$$

where

$$\Psi = \left(1 - \frac{1}{2}\epsilon^2 + \mathcal{O}(\epsilon^3)\right) t.$$

This solution is valid for all $t \geq 0$ since A_0 decays exponentially to its trivial rest point.

In [24], van der Pol developed a new method to obtain the asymptotic solution of the (since-called) van der Pol equation

$$\ddot{y} + y + \epsilon(y^2 - 1)\dot{y} = 0, \tag{2.25}$$

which he used to model electrical and physiological oscillations. We shall seek an asymptotic solution valid for $t = \mathcal{O}(\epsilon^{-1})$. When the method of multiple scales is applied, A_0 and ϕ_0 must satisfy

$$\frac{dA_0}{d\sigma} = \beta_1(A_0) = \frac{1}{2} A_0 \left(1 - \frac{1}{4} A_0^2\right), \quad A_0(0) = 1$$

and

$$\frac{d\phi_0}{d\sigma} = \frac{\alpha_1(A_0)}{A_0} = 0, \quad \phi_0(0) = 0.$$

This provides the unique limiting solution $y_0 = A_0(\sigma) \cos \Psi$, where $A_0(\sigma) = 2/\sqrt{1 + 3e^{-\sigma}}$ and $\Psi = \eta$, since $\phi_0(\sigma) = 0$. Thus y_0 uniformly approaches the periodic orbit $2 \cos \eta$ as $\sigma \to \infty$. Indeed, a *limit cycle* exists for all positive values of ϵ and all nontrivial initial values, though no exact formula for it is available, [25, Chapter 9].

At $\mathcal{O}(\epsilon)$, the coefficient A_1 must satisfy

$$\frac{d}{dA_0}\left(\frac{A_1(A_0)}{\beta_1(A_0)}\right) = 0, \quad A_1(1) = 0,$$

so $A_1(A_0) \equiv 0$. Moreover, B_1 must satisfy

$$\frac{d}{dA_0}\left(\frac{B_1(A_0)}{A_0}\right) = \frac{1}{\beta_1(A_0)}\left(\omega_0 - \frac{\mathcal{A}(A_0)}{A_0}\right), \quad B_1(1) = -\frac{3}{32},$$

where $\mathcal{A}(A_0) = -\frac{1}{8} A_0 + \frac{3}{16} A_0^3 - \frac{11}{256} A_0^5$. Since $A_0 \to 2$ as $\sigma \to \infty$, B_1 will remain nonsecular if and only if we choose

$$\omega_0 = -\frac{1}{16}.$$

The asymptotic solution to $\mathcal{O}(\epsilon)$ is then given by

$$y(t) = A_0 \cos \Psi + \epsilon\left[\left(-\frac{11}{64} A_0 - \frac{7}{64} A_0^3 + \frac{1}{8} A_0 \log(A_0)\right) \sin \Psi - \frac{1}{32} A_0^3 \sin 3\Psi\right] + \mathcal{O}(\epsilon^2),$$

where

$$\Psi = \left(1 - \frac{1}{16}\epsilon^2 + \mathcal{O}(\epsilon^3)\right) t.$$

The solution is asymptotically orbitally stable as $t \to \infty$, [25, Section 1.8], [26, Chapter 4].

3. Vector systems

In later sections, the starting point for the asymptotic methods presented is the system

$$\dot{\mathbf{x}} = \epsilon \mathbf{F}(\mathbf{x}, t, \epsilon), \tag{3.1}$$

in *periodic standard form*, with $\mathbf{x}(0)$ given. Here, $\mathbf{F}$ is smooth and 2π-periodic in t, with a power-series expansion in ϵ. Since the standard form (3.1) is central to what follows, we briefly explore its properties.

Consider the weakly nonlinear vector system

$$\dot{\mathbf{z}} = M\mathbf{z} + \epsilon \mathbf{N}(\mathbf{z}, t, \epsilon), \tag{3.2}$$

where the constant matrix M has purely imaginary, integral eigenvalues and a complete set of eigenvectors. We also assume that $\mathbf{N}$ is 2π-periodic in t. The exponential matrix $\exp(Mt)$ then has period 2π, so setting $\mathbf{z} = \exp(Mt)\mathbf{x}$ transforms (3.2) into (3.1), with the 2π-periodic forcing

$$\mathbf{F}(\mathbf{x}, t, \epsilon) \equiv \exp(-Mt)\,\mathbf{N}\,(\exp(Mt)\mathbf{x}, t, \epsilon).$$

We note that [27] and [28] solve (3.2) by an invariance condition method, while [29] shows that the Navier-Stokes equation for a slightly compressible fluid can be considered as an abstract equation of this form. A more general standard form than (3.1) is needed, however, to encounter the classic small-divisor problem [30, Chapter 5], [31, Chapters 2–3].

For the single weakly, nonlinear oscillator (1.1), one may easily obtain a system of the form (3.2) by letting $\dot{y} = x$, so that $\dot{x} = -y - \epsilon f(y, x)$. Then $\mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. A more practical choice, however, is to introduce polar coordinates (ρ, ϕ) by letting $y = \rho \cos(t + \phi)$ and $\dot{y} = -\rho \sin(t + \phi)$. In Sections 4 and 6 we will see how this choice leads to a much simpler system for variation on the slow timescale. Specifically, the slow systems will always be triangular, with the derivatives of the radius and phase depending only on the radius.

The regular perturbation process can be applied to initial-value problems for vector systems in the standard form (3.1). To obtain the regular perturbation series, set

$$\mathbf{x}_\epsilon(t) = \mathbf{x}_0(t) + \epsilon \mathbf{x}_1(t) + \epsilon^2 \mathbf{x}_2(t) + \cdots \tag{3.3}$$

and use the analogous series for the derivative. We naturally expand $\mathbf{F}(\mathbf{x}, t, \epsilon)$ in its Maclaurin expansion

$$\mathbf{F}(\mathbf{x}, t, \epsilon) = \mathbf{F}(\mathbf{x}_0, t, 0) + \epsilon\left[(\mathbf{x}_1 \cdot \bigtriangledown)\,\mathbf{F}(\mathbf{x}_0, t, 0) + \mathbf{F}_\epsilon(\mathbf{x}_0, t, 0)\right] + \cdots$$

and equate coefficients of like powers of ϵ in the differential system and initial value. We find that the vector coefficients $\mathbf{x}_j$ must, in turn, satisfy the linear systems

$$\dot{\mathbf{x}}_0 = 0, \quad \mathbf{x}_0(0) = \mathbf{x}(0),$$
$$\dot{\mathbf{x}}_1 = \mathbf{F}(\mathbf{x}_0, t, 0), \quad \mathbf{x}_1(0) = 0,$$
$$\dot{\mathbf{x}}_2 = (\mathbf{x}_1 \cdot \bigtriangledown)\,\mathbf{F}(\mathbf{x}_0, t, 0) + \mathbf{F}_\epsilon(\mathbf{x}_0, t, 0), \quad \mathbf{x}_2(0) = 0,$$

and so on. Upon integration, we uniquely obtain $\mathbf{x}_0(t) = \mathbf{x}(0)$,

$$\mathbf{x}_1(t) = \int_0^t \mathbf{F}(\mathbf{x}(0), s, 0)\mathrm{d}s,$$

and

$$\mathbf{x}_2(t) = \int_0^t \left[(\mathbf{x}_1(s) \cdot \bigtriangledown)\,\mathbf{F}(\mathbf{x}(0), s, 0) + \mathbf{F}_\epsilon(\mathbf{x}(0), s, 0)\right]\mathrm{d}s.$$

The procedure can be continued indefinitely and the resulting series will converge, like the so-called *matrizant*, [32, p. 63], to the unique solution of the initial-value problem on any finite t interval, for ϵ sufficiently small.

Naturally, resonance causes complications on long time intervals. Undetermined coefficients shows that any $\mathbf{x}_j(t)$ will have polynomial blowup as $t \to \infty$ whenever $\dot{\mathbf{x}}_j$ is a polynomial in t. A more complete understanding can be achieved since the assumed periodicity of $\mathbf{F}$ with respect to t allows us to employ Fourier series to show by induction that the term $\mathbf{x}_j(t)$ can generally grow like a polynomial in t of degree j. The constancy of $\mathbf{x}_0$, in particular,

shows that $\mathbf{x}_1(t) = a_0 t + \int_0^t (\mathbf{F}(\mathbf{x}(0), s, 0) - a_0)\, ds$ will grow linearly in t if and only if the constant term, a_0, in the Fourier expansion of $\mathbf{F}(\mathbf{x}(0), t, 0)$ is nonzero. (The integral is bounded since its integrand is periodic with a zero average.) Conditions guaranteeing the nonsecular behavior of later terms follow analogously.

One way to achieve solutions of (3.1) on longer time intervals is to apply the multiple scale ideas of Section 2. We therefore seek a generalized asymptotic expansion

$$\mathbf{x}_\epsilon(t, \sigma) = \mathbf{x}_0(t, \sigma) + \epsilon \mathbf{x}_1(t, \sigma) + \cdots, \tag{3.4}$$

where $\sigma = \epsilon t$ is a slow time. Here, it is not necessary to strain the fast time, t, to get an asymptotic solution for $t = \mathcal{O}(\epsilon^{-1})$. Since the chain rule implies that

$$\dot{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial t} + \epsilon \frac{\partial \mathbf{x}}{\partial \sigma},$$

we shall determine successive terms of the series (3.4) by equating coefficients of like powers of ϵ in the resulting partial differential equation

$$\frac{\partial \mathbf{x}}{\partial t} = \epsilon \left(\mathbf{F}(\mathbf{x}(t, \sigma, \epsilon), t, \epsilon) - \frac{\partial \mathbf{x}}{\partial \sigma} \right). \tag{3.5}$$

Thus, the first two equations

$$\frac{\partial \mathbf{x}_0}{\partial t} = 0, \quad \frac{\partial \mathbf{x}_1}{\partial t} = \mathbf{F}(\mathbf{x}_0, t, 0) - \frac{\partial \mathbf{x}_0}{\partial \sigma}$$

imply that the limiting solution

$$\mathbf{x}_0(t, \sigma) = \mathbf{A}_0(\sigma) \tag{3.6}$$

is independent of t, while its first correction $\mathbf{x}_1$ must satisfy

$$\frac{\partial \mathbf{x}_1}{\partial t} = \mathbf{F}(\mathbf{A}_0(\sigma), t, 0) - \frac{d\mathbf{A}_0}{d\sigma}. \tag{3.7}$$

Because $\mathbf{F}$ is a 2π-periodic function of t, $\frac{\partial \mathbf{x}_1}{\partial t}$ must have a zero average with respect to t in order for $\mathbf{x}_1$ to remain bounded as $t \to \infty$. Thus, the limiting solution $\mathbf{A}_0$ must satisfy the nonlinear initial value problem

$$\frac{d\mathbf{A}_0}{d\sigma} = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{F}(\mathbf{A}_0(\sigma), s, 0)\, ds, \quad \mathbf{A}_0(0) = \mathbf{x}(0), \tag{3.8}$$

a significant result which is traditionally obtained via the method of averaging. Due to the guaranteed local existence of $\mathbf{A}_0$, we can integrate (3.7) using the trivial initial value to obtain

$$\mathbf{x}_1(t, \sigma) = \frac{1}{2\pi} \int_0^t \left(\int_0^{2\pi} [F(A_0(\sigma), r, 0) - F(A_0(\sigma), s, 0)]\, ds \right) dr \tag{3.9}$$

as a function of $\mathbf{A}_0(\sigma)$ and t. Later terms in the series (3.4) follow in a straightforward fashion and the resulting approximation will be valid for $t = \mathcal{O}(\epsilon^{-1})$.

4. Averaging

The method of *averaging* was largely developed by Krylov and Bogoliubov in Kiev in the 1930's, [5, Appendix 7], [33], [34, Chapter 11] and the Ukrainian school continues to produce related asymptotics, [31, 35]. Morrison, [21], showed that, for the class of nonlinear oscillators (1.1) being considered, the methods of multiple scales and averaging produce the same results on the $\mathcal{O}(\epsilon^{-1})$ timescale (see also [36, 37]). Since the asymptotic validity of averaging had been rigorously proven (see *e.g.* [33, Chapter 6]), this validated the efficacy of multiple scales. As mentioned before, an independent proof of the asymptotic validity of multiple scales was given later in [23]. This connection between averaging and multiple scales was historically important for applied mathematicians because, although averaging had been rigorously justified, it had been applied only to a relatively small class of problems. Multiple scales, in contrast, had been applied to a much wider class of problems, including partial differential equations, though it had not been proven to be correct. To this day, averaging is largely used for systems of ordinary differential equations, although *homogenization* is a natural extension of averaging for PDEs, [38].

The method of averaging is traditionally applied to vector systems in a standard form such as (3.1). We shall first reduce the autonomous scalar equation (1.1) to this standard form by making a change to polar coordinates. To this end we let

$$y = \rho \cos(t + \phi) \quad \text{and} \quad \dot{y} = -\rho \sin(t + \phi),$$

where ρ and ϕ vary with t. The consistency condition $\frac{d}{dt} y = \dot{y}$ requires

$$\dot{\rho} \cos(t + \phi) - \rho \dot{\phi} \sin(t + \phi) = 0.$$

Moreover, substituting in (1.1) requires

$$\dot{\rho} \sin(t + \phi) + \rho \dot{\phi} \cos(t + \phi) = \epsilon f(\rho \cos(t + \phi), -\rho \sin(t + \phi))$$

Solving this linear system for $\dot{\rho}$ and $\rho \dot{\phi}$, shows that the two-vector

$$\mathbf{x} = \begin{pmatrix} \rho \\ \phi \end{pmatrix}$$

satisfies a planar system of the form (3.1) with

$$\mathbf{F}(\mathbf{x}, t, \epsilon) \equiv \begin{pmatrix} \sin(t + \phi) \\ \frac{1}{\rho} \cos(t + \phi) \end{pmatrix} f(\rho \cos(t + \phi), -\rho \sin(t + \phi)).$$

As shown in Section 3, the leading-order solution of (3.1) is simply the initial value $\mathbf{x}(0)$. This solution is valid for $\mathcal{O}(1)$ times, but is inappropriate for longer times. To achieve an asymptotic solution valid on $\mathcal{O}(\epsilon^{-1})$ timescales, we replace the constant $\mathbf{x}(0)$ with a vector

$$\mathbf{A}(\sigma) = \begin{pmatrix} R(\sigma) \\ \Phi(\sigma) \end{pmatrix}$$

that varies on the slow timescale $\sigma = \epsilon t$. At higher orders, there remains the potential for variation on the fast scale, so we also add a correction term,

$$\mathbf{U}(\mathbf{A}, t, \epsilon) = \begin{pmatrix} V(\mathbf{A}(\sigma), t, \epsilon) \\ W(\mathbf{A}(\sigma), t, \epsilon) \end{pmatrix},$$

to account for this. Thus, we shall employ the *near-identity* transformation

$$\mathbf{x} = \mathbf{A} + \epsilon\mathbf{U}(\mathbf{A}, t, \epsilon), \tag{4.1}$$

selecting the terms of the power-series expansion

$$\mathbf{U}(\mathbf{A}, t, \epsilon) \sim \mathbf{U}_0(\mathbf{A}, t) + \epsilon\mathbf{U}_1(\mathbf{A}, t) + \cdots$$

for the scaled correction $\mathbf{U}$ to be 2π periodic in t. We will thereby determine an appropriate smooth power series $\mathbf{M}(\mathbf{A}, \epsilon)$ in ϵ so that the slowly varying vector $\mathbf{A}(\sigma)$ will satisfy an autonomous *averaged* system

$$\dot{\mathbf{A}} = \epsilon\mathbf{M}(\mathbf{A}, \epsilon), \quad \mathbf{A}(0) = \mathbf{x}(0). \tag{4.2}$$

We can rewrite this as

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}\sigma} = \mathbf{M}(\mathbf{A}, \epsilon) \sim \mathbf{M}_0(\mathbf{A}) + \epsilon\mathbf{M}_1(\mathbf{A}) + \cdots, \tag{4.3}$$

so averaging integrates away the fast time scale in (3.1). We do all this in the hope that finding the asymptotic solution of (4.3) is a simpler challenge than integrating the original initial-value problem for large times. In particular, local existence of $\mathbf{A}$ on some $\sigma > 0$ interval is guaranteed and a regular perturbation procedure with respect to ϵ might suffice as a first attempt to obtain it asymptotically. Such near-identity transformations generalize a classical asymptotic procedure of von Ziepel [18, Chapter 5], [30, Chapter 5] (see also [39], which validates the procedure in terms of Gevrey asymptotics).

Applying the chain rule to the transformation (4.1) implies that

$$\dot{\mathbf{x}} = \epsilon\left[\left(\mathbf{I} + \epsilon\frac{\partial\mathbf{U}}{\partial\mathbf{A}}\right)\mathbf{M}(\mathbf{A}, \epsilon) + \frac{\partial\mathbf{U}}{\partial t}\right] = \epsilon\mathbf{F}(\mathbf{A} + \epsilon\mathbf{U}, t, \epsilon),$$

so the scaled correction $\mathbf{U}$ must satisfy the vector system

$$\frac{\partial\mathbf{U}}{\partial t} = \mathbf{F}(\mathbf{A} + \epsilon\mathbf{U}, t, \epsilon) - \left(\mathbf{I} + \epsilon\frac{\partial\mathbf{U}}{\partial\mathbf{A}}(\mathbf{A}, t, \epsilon)\right)\mathbf{M}(\mathbf{A}, \epsilon). \tag{4.4}$$

Because we require $\mathbf{U}$ to be periodic in t, we must pick $\mathbf{M}$ to make $\frac{\partial\mathbf{U}}{\partial t}$ have a zero average. Using the traditional notation (2.10), we may decompose any 2π-periodic function $g(t)$ into its average part, $\langle g(t)\rangle$, and its mean-free oscillatory remainder

$$\{g(t)\} \equiv \frac{1}{2\pi}\int_0^{2\pi}(g(t) - g(s))\,\mathrm{d}s = g(t) - \langle g(t)\rangle. \tag{4.5}$$

Thus, $\left\langle\frac{\partial\mathbf{U}}{\partial t}\right\rangle = 0$ and the balance of average terms in (4.4) determines the previously unspecified, t-independent forcing

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}\sigma} = \mathbf{M}(\mathbf{A}, \epsilon) \equiv \left(\mathbf{I} + \epsilon\left\langle\frac{\partial\mathbf{U}}{\partial\mathbf{A}}(\mathbf{A}, t, \epsilon)\right\rangle\right)^{-1}\langle\mathbf{F}(\mathbf{A} + \epsilon\mathbf{U}, t, \epsilon)\rangle \tag{4.6}$$

in (4.2). Having so balanced the average terms, we balance the remaining oscillatory terms (4.4) through the equation

$$\frac{\partial\mathbf{U}}{\partial t} = \{\mathbf{F}(\mathbf{A} + \epsilon\mathbf{U}, t, \epsilon)\} - \epsilon\left\{\frac{\partial\mathbf{U}}{\partial\mathbf{A}}\right\}\left(\mathbf{I} + \epsilon\left\langle\frac{\partial\mathbf{U}}{\partial\mathbf{A}}\right\rangle\right)^{-1}\langle\mathbf{F}(\mathbf{A} + \epsilon\mathbf{U}, t, \epsilon)\rangle, \tag{4.7}$$

which needs to be integrated to obtain the correction $\mathbf{U}$.

Progress now follows readily by simultaneously iterating termwise in (4.3) and (4.7). First, expanding (4.6) for small ϵ, we find that the first two forcing terms of the averaged equation (4.3) are

$$\mathbf{M}_0(\mathbf{A}) = \langle \mathbf{F}(\mathbf{A}, t, 0) \rangle$$

and

$$\mathbf{M}_1(\mathbf{A}) = \langle (\mathbf{U}_0 \cdot \triangledown) \, \mathbf{F}(\mathbf{A}, t, 0) + \mathbf{F}_\epsilon(\mathbf{A}, t, 0) \rangle - \left\langle \frac{\partial \mathbf{U}_0}{\partial \mathbf{A}} \right\rangle \langle \mathbf{F}(\mathbf{A}, t, 0) \rangle .$$

While $\mathbf{M}_0$ is completely specified, $\mathbf{M}_1$ depends on $\mathbf{U}_0$, so we must calculate $\mathbf{U}_0$ to determine $\mathbf{M}_1$. To find $\mathbf{U}$, we expand (4.7) for small ϵ and integrate. The first term immediately gives the periodic function

$$\mathbf{U}_0(\mathbf{A}, t) = \int_0^t \{ \mathbf{F}(A, s, 0) \} \, \mathrm{d}s. \tag{4.8}$$

Thus the $\mathcal{O}(\epsilon)$ solution to (3.1) is

$$\mathbf{x}(t) = \mathbf{A}(\sigma) + \epsilon \mathbf{U}_0(\mathbf{A}, t) + \mathcal{O}(\epsilon^2), \tag{4.9}$$

where the slowly varying function $\mathbf{A}(\sigma)$ must still be determined by solving the approximate averaged problem

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}\sigma} = \mathbf{M}_0(\mathbf{A}) + \epsilon \mathbf{M}_1(\mathbf{A}) + \mathcal{O}(\epsilon^2), \quad \mathbf{A}(0) = \mathbf{x}(0). \tag{4.10}$$

This procedure may easily be extended to higher orders in ϵ.

To solve the averaged problem (4.10), we try a regular power series

$$\mathbf{A}(\sigma) = \mathbf{A}_0(\sigma) + \epsilon \mathbf{A}_1(\sigma) + \mathcal{O}(\epsilon^2). \tag{4.11}$$

The leading term satisfies the nonlinear initial-value problem

$$\frac{\mathrm{d}\mathbf{A}_0}{\mathrm{d}\sigma} = \mathbf{M}_0(\mathbf{A}_0), \quad \mathbf{A}_0(0) = \mathbf{x}(0),$$

while the second term will satisfy the linearized problem

$$\frac{\mathrm{d}\mathbf{A}_1}{\mathrm{d}\sigma} = \frac{\mathrm{d}\mathbf{M}_0}{\mathrm{d}\mathbf{A}}(\mathbf{A}_0) \, \mathbf{A}_1 + \mathbf{M}_1(\mathbf{A}_0), \quad \mathbf{A}_1(0) = 0.$$

In particular, as with multiple scales, the term $\mathbf{M}_0$ for the oscillator (1.1) is given in terms of the Fourier coefficients of $f(y, \dot{y})$, *viz.*

$$\mathbf{M}_0(\mathbf{A}_0) = \langle \mathbf{F}(\mathbf{A}_0, t, 0) \rangle = \begin{pmatrix} \beta_1(\mathbf{A}_0) \\ \frac{\alpha_1(\mathbf{A}_0)}{\mathbf{A}_0} \end{pmatrix}. \tag{4.12}$$

We now illustrate these procedures.

Recall the Duffing equation (1.7). The first-order system that results from the change to polar coordinates is

$$\begin{pmatrix} \dot{\rho} \\ \dot{\phi} \end{pmatrix} = \epsilon \begin{pmatrix} \frac{1}{4}\rho^3 \sin(2(t+\phi)) + \frac{1}{8}\rho^3 \sin(4(t+\phi)) \\ \frac{3}{8}\rho^2 + \frac{1}{2}\rho^2 \cos(2(t+\phi)) + \frac{1}{8}\rho^2 \cos(4(t+\phi)) \end{pmatrix}.$$

Substituting the near-identity transformation (4.1) and averaging, provides

$$\frac{dR}{d\sigma} = \mathcal{O}(\epsilon^2) \quad \text{and} \quad \frac{d\Phi}{d\sigma} = \frac{3}{8}R^2 - \epsilon \frac{21}{256}R^2 + \mathcal{O}(\epsilon^2),$$

along with the first correction term

$$\mathbf{U}_0(\mathbf{A}, t) = \begin{pmatrix} \frac{5}{32}R^2 - \frac{1}{8}R^3\cos(2(t+\Phi)) - \frac{1}{32}R^3\cos(4(t+\Phi)) \\ \frac{1}{4}R^2\sin(2(t+\Phi)) + \frac{1}{32}R^2\sin(4(t+\Phi)) \end{pmatrix}.$$

Using the initial values $R(0) = 1$ and $\Phi(0) = 0$, we find

$$R(\sigma) = 1 + \mathcal{O}(\epsilon^2\sigma) \quad \text{and} \quad \Phi(\sigma) = \left(\frac{3}{8} - \frac{21}{256}\epsilon\right)\sigma + \mathcal{O}(\epsilon^2\sigma).$$

Thus the solution in polar coordinates for bounded σ is

$$\rho = 1 + \epsilon\left(\frac{5}{32} - \frac{1}{8}\cos(2(t+\Phi)) - \frac{1}{32}\cos(4(t+\Phi))\right) + \mathcal{O}(\epsilon^2),$$

$$\phi = \Phi + \epsilon\left(\frac{1}{4}\sin(2(t+\Phi)) + \frac{1}{32}\sin(4(t+\Phi))\right) + \mathcal{O}(\epsilon^2).$$

Inserting these results in $y = \rho\cos(t+\phi)$ provides the solution in rectangular coordinates as

$$y(t) = \cos(t+\Phi) + \frac{\epsilon}{32}\left(\cos(3(t+\Phi)) - \cos(t+\Phi)\right) + \mathcal{O}(\epsilon^2),$$

where the phase satisfies

$$t + \Phi = \left(1 + \frac{3}{8}\epsilon - \frac{21}{256}\epsilon^2 + \mathcal{O}(\epsilon^3)\right)t.$$

This approximation, valid for finite σ, exactly matches the one we determined using multiple scales.

As a second example, take the damped linear oscillator (1.15). The first-order system in polar coordinates is

$$\begin{pmatrix} \dot{\rho} \\ \dot{\phi} \end{pmatrix} = \epsilon\begin{pmatrix} -\rho + \rho\cos(2(t+\phi)) \\ -\sin(2(t+\phi)) \end{pmatrix}.$$

Substituting the near-identity transformation (4.1) and averaging, we obtain

$$\frac{dR}{d\sigma} = -R + \mathcal{O}(\epsilon^2) \quad \text{and} \quad \frac{d\Phi}{d\sigma} = -\frac{1}{2}\epsilon + \mathcal{O}(\epsilon^2).$$

We also find the first correction term

$$\mathbf{U}_0(\mathbf{A}, t) = \begin{pmatrix} \frac{1}{2}R\sin(2(t+\Phi)) \\ \frac{1}{2}(1 - \cos(2(t+\Phi))) \end{pmatrix}.$$

Using $R(0) = 1$ and $\Phi(0) = 0$, we find

$$R(\sigma) = e^{-\sigma} + \mathcal{O}(\epsilon^2\sigma) \quad \text{and} \quad \Phi(\sigma) = -\frac{1}{2}\epsilon\sigma + \mathcal{O}(\epsilon^2\sigma).$$

Thus the solution for bounded σ is

$$\rho = R + \frac{\epsilon}{2} R \sin(2(t+\Phi)) + \mathcal{O}(\epsilon^2),$$

$$\phi = \Phi + \frac{\epsilon}{2}(1 - \cos(2(t+\Phi))) + \mathcal{O}(\epsilon^2).$$

Inserting into $y = \rho \cos(t+\phi)$, we obtain

$$y(t) = e^{-\sigma} \cos(t+\Phi) + \epsilon e^{-\sigma} \sin(t+\Phi) + \mathcal{O}(\epsilon^2),$$

where

$$t + \Phi = \left(1 - \frac{1}{2}\epsilon^2 + \mathcal{O}(\epsilon^3)\right) t.$$

This approximation again exactly matches the one determined with multiple scales.

As a third example, consider the van der Pol equation (2.25). The first-order system in polar coordinates is

$$\begin{pmatrix} \dot{\rho} \\ \dot{\phi} \end{pmatrix} = \epsilon \begin{pmatrix} -\frac{1}{8}\rho^2 + \frac{1}{2}\rho - \frac{1}{2}\rho \cos(2(t+\phi)) + \frac{1}{8}\rho^3 \cos(4(t+\phi)) \\ \left(\frac{1}{2} - \frac{1}{4}\rho^2\right)\sin(2(t+\phi)) - \frac{1}{8}\rho^2 \sin(4(t+\phi)) \end{pmatrix}.$$

Using the near-identity transformation (4.1) and averaging, we determine

$$\frac{dR}{d\sigma} = \frac{1}{2}R\left(1 - \frac{1}{4}R^2\right) + \mathcal{O}(\epsilon^2),$$

$$\frac{d\Phi}{d\sigma} = \epsilon\left(-\frac{1}{8} + \frac{11}{32}R^2 - \frac{21}{256}R^4\right) + \mathcal{O}(\epsilon^2),$$

and the first correction term

$$\mathbf{U}_0(\mathbf{A}, t) = \begin{pmatrix} -\frac{1}{4}R\sin(2(t+\Phi)) + \frac{1}{32}R^3 \sin(4(t+\Phi)) \\ \frac{1}{4} - \frac{5}{32}R^2 - \frac{1}{4}\left(1 - \frac{1}{2}R^2\right)\cos(2(t+\Phi)) + \frac{1}{32}R^2 \cos(4(t+\Phi)) \end{pmatrix}.$$

We let

$$R(\sigma) = R_0(\sigma) + \epsilon R_1(\sigma) + \mathcal{O}(\epsilon^2) \quad \text{and} \quad \Phi(\sigma) = \Phi_0(\sigma) + \epsilon \Phi_1(\sigma) + \mathcal{O}(\epsilon^2),$$

for $R_0(0) = 1$ and $R_1(0) = \Phi_0(0) = \Phi_1(0) = 0$. The resulting leading-order amplitude is

$$R_0(\sigma) = \frac{2}{\sqrt{1 + 3e^{-\sigma}}},$$

while $R_1(\sigma)$ is trivial. Similarly, the leading-order phase is $\Phi_0(\sigma) \equiv 0$. Thus, the solution for bounded σ is

$$\rho = R_0 + \epsilon\left(-\frac{1}{4}R_0 \sin(2(t+\epsilon\Phi_1)) + \frac{1}{32}R_0^3 \sin(4(t+\epsilon\Phi_1))\right) + \mathcal{O}(\epsilon^2),$$

$$\phi = \epsilon\left(\Phi_1 + \frac{1}{4} - \frac{5}{32}R_0^2 - \frac{1}{4}\left(1 - \frac{1}{2}R_0^2\right)\cos(2(t+\epsilon\Phi_1)) + \frac{1}{32}R_0^2 \cos(4(t+\epsilon\Phi_1))\right) + \mathcal{O}(\epsilon^2).$$

Now the amplitude, R_0, tends to its asymptotically stable rest point $R_0(\infty) = 2$, as $\sigma \to \infty$. As a result, the forcing in the differential equation for Φ_1 tends to a constant and the phase will tend to a linear function of σ. We now split $\Phi_1(\sigma)$ into the sum of its unbounded part, $\hat{\Phi}_1$, and a bounded correction $\tilde{\Phi}_1$. Substituting $R = R_0(\infty)$ in the equation for the phase, $\hat{\Phi}_1$ must satisfy $\frac{d\hat{\Phi}_1}{d\sigma} = -\frac{1}{16}$, $\hat{\Phi}_1(0) = 0$, so the supplementary bounded part must satisfy $\frac{d\tilde{\Phi}_1}{d\sigma} = -\frac{1}{16} + \frac{11}{32} R_0^2 - \frac{21}{256} R_0^4$, $\tilde{\Phi}_1(0) = 0$. Simple integrations provide

$$\hat{\Phi}_1(\sigma) = -\frac{1}{16}\sigma \quad \text{and} \quad \tilde{\Phi}_1(\sigma) = \frac{21}{64}\left(R_0^2 - 1\right) - \frac{1}{8}\log(R_0).$$

To construct the approximate solution in rectangular coordinates, we shall retain $\hat{\Phi}_1$ in the arguments of the trigonometric functions, while Taylor-expanding $\tilde{\Phi}_1$ outside. In this way, the unbounded growth of the phase will not destroy the ordering of our generalized asymptotic expansion. Using the expansions for ρ and ϕ in $y = \rho\cos(t + \phi)$, the solution in rectangular coordinates is

$$y(t) = R_0 \cos\left(t + \epsilon\hat{\Phi}_1\right) + \epsilon\left[-\frac{1}{32}R_0^3 \sin\left(3\left(t + \epsilon\hat{\Phi}_1\right)\right)\right.$$

$$\left. + \left(-\frac{11}{64}R_0 - \frac{7}{64}R_0^3 + \frac{1}{8}R_0\log(R_0)\right)\sin\left(t + \epsilon\hat{\Phi}_1\right)\right] + \mathcal{O}(\epsilon^2)$$

with

$$t + \epsilon\hat{\Phi}_1 = \left(1 - \frac{1}{16}\epsilon^2 + \mathcal{O}(\epsilon^3)\right)t.$$

This exactly matches the result found using multiple scales.

As mentioned earlier, one attraction of the method of averaging is that it is proven to give correct asymptotic approximations on (at least) $\mathcal{O}(\epsilon^{-1})$ time intervals. We now provide an outline for such a proof. Given a system in the standard form (3.1), suppose one determines (through whatever means) an approximation,

$$\mathbf{x}_\epsilon^m(t) = \mathbf{x}_0(t) + \epsilon\mathbf{x}_1(t) + \cdots + \epsilon^m\mathbf{x}_m(t), \tag{4.13}$$

to the solution $\mathbf{x}(t)$. This approximation will in general satisfy a differential equation

$$\dot{\mathbf{x}}_\epsilon^m = \epsilon\mathbf{F}(\mathbf{x}_\epsilon^m, t, \epsilon) + \epsilon^{m+2}\mathcal{R}(t), \tag{4.14}$$

where the remainder term $\mathcal{R}(t)$ represents the error made by using (4.13) in place of the true solution. One may then replace (3.1) and (4.14) with the equivalent integral equations and subtract to find that

$$\left|\mathbf{x}(t) - \mathbf{x}_\epsilon^m(t)\right| \le \epsilon\int_0^t \left|\mathbf{F}(\mathbf{x}, s, \epsilon) - \mathbf{F}(\mathbf{x}_\epsilon^m, s, \epsilon)\right| ds + \epsilon^{m+2}\int_0^t \mathcal{R}(s)\,ds. \tag{4.15}$$

A Gronwall-inequality estimate will then imply that

$$\left|\mathbf{x}(t) - \mathbf{x}_\epsilon^m(t)\right| = \mathcal{O}\left(\epsilon^{m+2}t e^{\epsilon t}\right),$$

so if $0 \le t \le \frac{L}{\epsilon}$, where L is a constant independent of ϵ, we obtain

$$\left|\mathbf{x}(t) - \mathbf{x}_\epsilon^m(t)\right| = \mathcal{O}(\epsilon^{m+1}).$$

The technical difficulties in such a proof reside in the methodology for determining the approximation, $\mathbf{x}_\epsilon^m(t)$, and the estimate of the remainder, $\mathcal{R}(t)$. Since the proof is independent of the method of approximation, we can use it for any method that proceeds from the standard form (3.1), including the methods of averaging and multiple scales, and even the renormalization group method (to be discussed next). We refer readers interested in complete proofs to [5,33] for averaging, to [23] for multiple scales and to [40] for the renormalization group method.

5. Renormalization

Chen *et al.* [16], developed a *renormalization group* method as a unified tool for asymptotic analysis. In particular, they sought to show that the renormalization-group approach was more efficient and accurate than standard methods in extracting global information from the perturbation expansion. Though experienced readers will naturally judge the efficacy of the arguments for themselves, this paper is especially impressive due to the number and variety of examples considered. Much follow-up has occurred, especially by theoretical physicists, including the papers [41–43]. Connections between the renormalization group and normal forms are made in [40], while many applications (including some for PDEs) have been made in references [29, 44, 45].

   For the weakly nonlinear oscillator, (1.1), the renormalization-group method of [16] proceeds in three distinct stages. First, one obtains the regular perturbation (or *naive*) expansion. This naive expansion is parameterized by the initial amplitude and phase and will generally include unbounded (secular) terms as $t \to \infty$. Next, one uses near-identity transformations to replace the constant initial amplitude and phase by slowly varying functions satisfying the initial conditions. The correction built into the near-identity transformation is then used to remove secular terms from the naive expansion. The remaining secular-free (or *bare*) expansion is then potentially valid on a long timescale. We then determine a pair of *renormalized* equations for the amplitude and phase from the near-identity transformation by enforcing the constancy of the initial amplitude and phase values. Lastly, we solve the renormalized amplitude and phase equations asymptotically. This stage of the process ultimately determines the timescale over which the approximations are valid. We observe more generally that *renormalization*, somewhat like the neutrix calculus [46], involves "canceling divergences" [47]. We also observe that averaging results in slowly varying coefficients, as does Whitham's analysis for dispersive waves [48, Part 2].

   To generate the naive expansion, let

$$y_\epsilon(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots \tag{5.1}$$

be the regular perturbation solution of (1.1), using successive terms

$$y_0(t) = A_\epsilon \cos(t + \psi_\epsilon)$$

and

$$y_j(t) = \int_0^t \sin(t-s) g_{j-1}(y_0(s), \ldots, \dot{y}_{j-1}(s)) \mathrm{d}s$$

for each $j \geq 1$. We leave the amplitude A_ϵ and the phase shift ψ_ϵ as ϵ-dependent constants in the leading coefficient $y_0(t, A_\epsilon, \psi_\epsilon)$ to emphasize the dependence of later coefficients

$y_j(t, A_\epsilon, \psi_\epsilon)$ on two such constants of integration. The naive expansion will thus be determined as a generalized asymptotic expansion

$$y_\epsilon(t, A_\epsilon, \psi_\epsilon) = y_0(t, A_\epsilon, \psi_\epsilon) + \epsilon y_1(t, A_\epsilon, \psi_\epsilon) + \cdots. \tag{5.2}$$

Having determined the naive expansion, we use the near-identity transformations

$$A_\epsilon = A + \epsilon U(A, t, \epsilon), \tag{5.3}$$
$$\psi_\epsilon = \Psi + \epsilon V(A, t, \epsilon) \tag{5.4}$$

to approximately replace the constants A_ϵ and ψ_ϵ with slowly varying functions A and Ψ. The power-series expansions for the secular corrections U and V will be constructed termwise in ϵ to eliminate secular terms in the next coefficient of the resulting bare expansion

$$Y(t, A, \Psi, \epsilon) = y_\epsilon(t, A + \epsilon U, \Psi + \epsilon V) = Y_0(t, A, \Psi) + \epsilon Y_1(t, A, \Psi) + \cdots. \tag{5.5}$$

More generally, the secular corrections U and V in (5.3) and (5.4) would depend on Ψ as well, but for the restricted class of oscillators (1.1), experience suggests that dependence on only the amplitude A is an allowable simplification.

Since

$$y_0(t, A_\epsilon, \psi_\epsilon) = A_\epsilon \cos(t + \psi_\epsilon) \equiv \tilde{y}_0(t, A_\epsilon, \psi_\epsilon)$$

is secular-free, the leading term of the bare expansion is

$$Y_0(t, A, \Psi) = A \cos(t + \Psi). \tag{5.6}$$

Moreover, since Y_1 denotes the coefficient of ϵ in (5.5),

$$Y_1(t, A, \Psi) = y_1(t, A, \Psi) + \frac{\partial y_0}{\partial A_\epsilon}(t, A, \Psi) U_0(A, t) + \frac{\partial y_0}{\partial \psi_\epsilon}(t, A, \Psi) V_0(A, t).$$

Recall, however, that when we expose the secular terms of y_1, we get

$$y_1(t, A_\epsilon, \psi_\epsilon) = [\beta_1(A_\epsilon) \cos(t + \psi_\epsilon) - \alpha_1(A_\epsilon) \sin(t + \psi_\epsilon)] t + \tilde{y}_1(t, A_\epsilon, \psi_\epsilon)$$

where

$$\tilde{y}_1(t, A_\epsilon, \psi_\epsilon) \equiv -\int_0^t \sin(t - s) \tilde{f}(A_\epsilon \cos(s + \psi_\epsilon), -A_\epsilon \sin(s + \psi_\epsilon)) \mathrm{d}s$$

is secular-free since the first harmonics of f were split-off to define $\tilde{f}$. Thus, the second term

$$Y_1(t, A, \Psi) \equiv \tilde{y}_1(t, A, \Psi) \tag{5.7}$$

of the bare expansion (5.5) becomes secular-free when we select the first correction terms

$$U_0(t, A) = -\beta_1(A) t \quad \text{and} \quad V_0(t, A) = -\frac{\alpha_1(A)}{A} t$$

from (5.3), (5.4) to precisely cancel the coefficients of $\cos(t + \Psi)$ and $\sin(t + \Psi)$ in Y_1.

Analogously, we select U_1 and V_1 to cancel secular terms in y_2 to leave a secular-free Y_2 at $\mathcal{O}(\epsilon^2)$. Because the term y_2 of the naive expansion is generally a quadratic in t, it is natural to expect the corresponding correction terms U_1 and V_1 in (5.3), (5.4) to also be quadratic. In general, y_n will have secular terms containing powers of at most t^n, so we expect U_{n-1} and

V_{n-1} to also be polynomials in t of degree n. Continuing to choose the terms of the near-identity transformations to cancel secular terms, what results is an asymptotic expansion

$$Y(t, A, \Psi) = \tilde{y}_0(t, A, \Psi) + \epsilon \tilde{y}_1(t, A, \Psi) + \epsilon^2 \tilde{y}_2(t, A, \Psi) + \cdots, \tag{5.8}$$

determined by the nonsecular part $\tilde{y}(t, A, \Psi, \epsilon)$ of the naive expansion $y_\epsilon(t, A_\epsilon, \psi_\epsilon)$, where the constants A_ϵ and ψ_ϵ are replaced by slowly varying functions A and Ψ, [49].

Since A_ϵ and ψ_ϵ are constants, their time derivatives are zero. Enforcing these conditions provides a pair of evolution equations for the slowly-varying functions A and Ψ. Specifically, since $\frac{dA_\epsilon}{dt} = 0$, the chain rule provides the evolution equation

$$\frac{dA}{d\sigma} = -\left(1 + \epsilon \frac{\partial U}{\partial A}\right)^{-1} \frac{\partial U}{\partial t} \equiv h(A, \epsilon) \tag{5.9}$$

for the amplitude A, where we can expand

$$h(A, \epsilon) \equiv h_0(A) + \epsilon h_1(A) + \epsilon^2 h_2(A) + \cdots$$

as an *autonomous* forcing function. Similarly, $\frac{d\psi_\epsilon}{dt} = 0$ implies the evolution equation

$$\frac{d\Psi}{d\sigma} = -\frac{\partial V}{\partial t} - \epsilon \frac{\partial V}{\partial A} h(A, \epsilon) \equiv k(A, \epsilon) \tag{5.10}$$

for the slowly varying phase. Again, the forcing

$$k(A, \epsilon) \equiv k_0(A) + \epsilon k_1(A) + \epsilon^2 k_2(A) + \cdots$$

is autonomous. Equations (5.9) and (5.10) are, respectively, the renormalized amplitude and phase equations. These evolution equations are determined as asymptotic series, since we generally solve for U and V termwise. We likewise determine initial conditions for A and Ψ by asymptotically solving the equations

$$Y(0, A(0, \epsilon), \Psi(0, \epsilon)) = 1 \quad \text{and} \quad \dot{Y}(0, A(0, \epsilon), \Psi(0, \epsilon)) = 0$$

for the successive coefficients in

$$A(0, \epsilon) = a_0 + \epsilon a_1 + \epsilon^2 a_2 + \cdots \quad \text{and} \quad \Psi(0, \epsilon) = b_0 + \epsilon b_1 + \epsilon^2 b_2 + \cdots.$$

Not surprisingly, the limiting renormalized amplitude and phase equations coincide with the limiting equations previously found via multiple scales and averaging. In particular, in terms of the first-harmonic Fourier coefficients of $f(y_0, \dot{y}_0)$, $h_0(A) = \beta_1(A)$ and $k_0(A) = \frac{\alpha_1(A)}{A}$. The desired asymptotic solution of (1.1) follows from the bare expansion, (5.5), using asymptotic solutions, A and Ψ, of (5.9) and (5.10) wherever they are defined.

Since the initial-value problem for

$$\frac{dA}{d\sigma} = h(A, \epsilon), \tag{5.11}$$

is decoupled from that for the phase, we first seek its solution as a regular power series

$$A_\epsilon(\sigma) = A_0(\sigma) + \epsilon A_1(\sigma) + \cdots \tag{5.12}$$

in ϵ. The solution of the limiting renormalized equation

$$\frac{dA_0}{d\sigma} = \beta_1(A_0), \quad A_0(0) = a_0$$

is defined at least up to some finite σ. The correction term $A_1(\sigma)$ satisfies the linearized problem

$$\frac{\mathrm{d}A_1}{\mathrm{d}\sigma} = \frac{\mathrm{d}\beta_1}{\mathrm{d}A}(A_0)\,A_1 - h_1(A_0), \quad A_1(0) = a_1.$$

Since the Jacobian $\frac{\partial A_0}{\partial a_0}$ satisfies $\frac{\mathrm{d}J}{\mathrm{d}\sigma} = \frac{\mathrm{d}\beta_1}{\mathrm{d}A}(A_0)\,J$, we immediately determine

$$A_1(\sigma) = \frac{\partial A_0(\sigma)}{\partial a_0}\left(\left[\frac{\partial A_0}{\partial a_0}(0)\right]^{-1}a_1 - \int_0^\sigma \left[\frac{\partial A_0}{\partial a_0}(r)\right]^{-1}h_1(r)\,\mathrm{d}r\right), \tag{5.13}$$

as long as $A_0(\sigma)$ remains defined. Later terms follow uniquely in succession. If A_0 decays to an asymptotically stable rest point, all terms of the series (5.12) will also decay exponentially and the renormalized solution for the amplitude A will be valid asymptotically for all $t \geq 0$. Due to the triangular structure of the renormalized system, (5.10) immediately implies that

$$\Psi(\sigma, \epsilon) = \Psi(0, \epsilon) + \int_0^\sigma k(A(r), \epsilon)\mathrm{d}r. \tag{5.14}$$

In many cases, the solution for Ψ will contain terms that grow as $\sigma \to \infty$. Though these terms will not destabilize the amplitude, they can shift the phase on the $\mathcal{O}(\epsilon^{-1})$ timescale, thus limiting the uniform validity of the renormalization group solution to such times. We will now consider several examples.

First, let us return to the Duffing equation, (1.7). Renormalizing provides the bare approximation

$$Y(t, A, \Psi, \epsilon) = A\cos(t + \Psi) + \epsilon\frac{A^3}{32}\cos(3(t + \Psi)) + \mathcal{O}(\epsilon^2),$$

and the evolution equations

$$\frac{\mathrm{d}A}{\mathrm{d}\sigma} = \mathcal{O}(\epsilon^2) \quad \text{and} \quad \frac{\mathrm{d}\Psi}{\mathrm{d}\sigma} = \frac{3}{8}A^2 - \epsilon\frac{15}{256}A^4 + \mathcal{O}(\epsilon^2),$$

for the amplitude and phase. To find initial conditions, we must asymptotically solve the system

$$A(0, \epsilon)\cos(\Psi(0, \epsilon)) + \epsilon\frac{A^3(0, \epsilon)}{32}\cos(3\Psi(0, \epsilon)) + \mathcal{O}(\epsilon^2) = 1,$$

$$A(0, \epsilon)\sin(\Psi(0, \epsilon)) + \epsilon\frac{3A^3(0, \epsilon)}{8}\left(\sin(\Psi(0, \epsilon)) + \frac{1}{4}\sin(3\Psi(0, \epsilon))\right) + \mathcal{O}(\epsilon^2) = 0.$$

This determines $A(0, \epsilon) = 1 - \frac{\epsilon}{32} + \mathcal{O}(\epsilon^2)$ and $\Psi(0, \epsilon) = \mathcal{O}(\epsilon^2)$, so $A(\sigma) = 1 - \frac{\epsilon}{32} + \mathcal{O}(\epsilon^2)\mathcal{O}(\epsilon^2\sigma)$ and $\Psi(\sigma) = \left(\frac{3}{8} - \epsilon\frac{21}{256}\right)\sigma + \mathcal{O}(\epsilon^2)\mathcal{O}(\epsilon^2\sigma)$. Thus,

$$y(t) = \cos(t + \Psi) + \frac{\epsilon}{32}\left(\cos(3(t + \Psi)) - \cos(t + \Psi)\right) + \mathcal{O}(\epsilon^2),$$

for the phase

$$t + \Psi = \left(1 + \frac{3}{8}\epsilon - \frac{21}{256}\epsilon^2 + \mathcal{O}(\epsilon^3)\right)t.$$

This expansion agrees exactly with the previous results from multiple scales and averaging.

Now reconsider the weakly damped linear oscillator (1.15). Renormalization provides the bare approximation

$$Y(t, A, \Psi, \epsilon) = A \cos(t + \Psi) + \mathcal{O}(\epsilon^2),$$

and the renormalized equations

$$\frac{dA}{d\sigma} = -A + \mathcal{O}(\epsilon^2) \quad \text{and} \quad \frac{d\Psi}{d\sigma} = -\frac{1}{2}\epsilon + \mathcal{O}(\epsilon^2).$$

To find initial conditions, we asymptotically solve

$$A(0, \epsilon) \cos(\Psi(0, \epsilon)) + \mathcal{O}(\epsilon^2) = 1,$$

$$-A(0, \epsilon) \sin(\Psi(0, \epsilon)) - \epsilon A(0, \epsilon) \cos(\Psi(0, \epsilon)) + \mathcal{O}(\epsilon^2) = 0$$

to obtain $A(0, \epsilon) = 1 + \mathcal{O}(\epsilon^2)$ and $\Psi(0, \epsilon) = -\epsilon + \mathcal{O}(\epsilon^2)$. Thus, $A(\sigma) = e^{-\sigma} + \mathcal{O}(\epsilon^2)\mathcal{O}(\epsilon^2\sigma)$. The leading-order phase is identically zero, but since A decays to the trivial asymptotically stable rest point, we will use the splitting technique (as we did for the van der Pol equation) to determine the next term Ψ_1. When we used averaging, we did no splitting because both the initial values and the rest point were trivial. Now, however, the nontrivial initial value generates a nonzero bounded contribution to Ψ_1. As before, let $\hat{\Psi}_1$ be the linear, unbounded part and $\tilde{\Psi}_1$ the bounded part of Ψ_1. Such a splitting implies the two supplementary initial-value problems

$$\frac{d\hat{\Psi}_1}{d\sigma} = -\frac{1}{2}, \quad \hat{\Psi}_1(0) = 0 \quad \text{and} \quad \frac{d\tilde{\Psi}_1}{d\sigma} = 0, \quad \tilde{\Psi}_1(0) = -1,$$

with solutions $\hat{\Psi}_1(\sigma) = -\frac{1}{2}\sigma$ and $\tilde{\Psi}_1(\sigma) = -1$. Thus the asymptotic solution is

$$y(t) = e^{-\sigma} \cos(t + \hat{\Psi}_1) - \epsilon e^{-\sigma} \sin(t + \hat{\Psi}_1) + \mathcal{O}(\epsilon^2),$$

for the phase

$$t + \hat{\Psi}_1 = \left(1 - \frac{1}{2}\epsilon^2 + \mathcal{O}(\epsilon^3)\right) t,$$

in exact agreement with the multiple scales and averaging results.

As a third example, let us return to the van der Pol equation (2.25). Renormalization provides the bare approximation

$$Y(t, A, \Psi, \epsilon) = A \cos(t + \Psi) + \epsilon \frac{A^3}{32} \sin(3(t + \Psi)) + \mathcal{O}(\epsilon^2)$$

and the renormalized equations

$$\frac{dA}{d\sigma} = \frac{1}{2} A \left(1 - \frac{1}{4}A^2\right) + \mathcal{O}(\epsilon^2),$$

$$\frac{d\Psi}{d\sigma} = \epsilon \left(-\frac{1}{8} + \frac{1}{8}A^2 - \frac{7}{256}A^4\right) + \mathcal{O}(\epsilon^2)$$

for the amplitude and phase. To find initial conditions, we must asymptotically solve

$$A(0, \epsilon) \cos(\Psi(0, \epsilon)) - \epsilon \frac{A^3(0, \epsilon)}{32} \sin(\Psi(0, \epsilon)) + \mathcal{O}(\epsilon^2) = 1,$$

$$-A(0, \epsilon) \sin(\Psi(0, \epsilon)) + \epsilon \left(-\frac{3A^3(0, \epsilon)}{32} \cos(3\Psi(0, \epsilon)) \right.$$

$$\left. + \frac{A(0, \epsilon)}{2} \left(1 - \frac{A^2(0, \epsilon)}{4} \right) \cos(\Psi(0, \epsilon)) \right) + \mathcal{O}(\epsilon^2) = 0$$

to obtain $A(0, \epsilon) = 1 + \mathcal{O}(\epsilon^2)$ and $\Psi(0, \epsilon) = \frac{9}{32}\epsilon + \mathcal{O}(\epsilon^2)$. We find

$$A(\sigma) = A_0(\sigma) + \mathcal{O}(\epsilon^2)\mathcal{O}(\epsilon^2\sigma) \equiv \frac{2}{\sqrt{1 + 3e^{-\sigma}}} + \mathcal{O}(\epsilon^2) + \mathcal{O}(\epsilon^2\sigma).$$

The leading-order amplitude, A_0, has the asymptotically stable rest point $A_0(\infty) = 2$. As usual, the leading-order contribution of the phase is $\Psi_0(\sigma) \equiv 0$, but because we have both a nontrivial rest point for the amplitude and a nonzero initial phase, we split Ψ_1 into bounded and unbounded parts. This splitting results in

$$\frac{d\hat{\Psi}_1}{d\sigma} = -\frac{1}{16}, \quad \hat{\Psi}_1(0) = 0$$

and

$$\frac{d\tilde{\Psi}_1}{d\sigma} = -\frac{1}{16} + \frac{1}{8}A_0^2 - \frac{7}{256}A_0^4, \quad \tilde{\Psi}_1(0) = \frac{9}{32},$$

with solutions $\hat{\Psi}_1(\sigma) = -\frac{1}{16}\sigma$ and $\tilde{\Psi}_1 = \frac{11}{64} + \frac{7}{64}A_0^2 - \frac{1}{8}\log(A_0)$. The resulting solution,

$$y(t) = A_0 \cos\left(t + \epsilon\hat{\Psi}_1\right) + \epsilon \left[-\frac{1}{32}A_0^3 \sin\left(3\left(t + \epsilon\hat{\Psi}_1\right)\right) \right.$$

$$\left. + \left(-\frac{11}{64}A_0 - \frac{7}{64}A_0^3 + \frac{1}{8}A_0\log(A_0) \right) \sin\left(t + \epsilon\hat{\Psi}_1\right) \right] + \mathcal{O}(\epsilon^2)$$

for the phase

$$t + \epsilon\hat{\Psi}_1 = \left(1 - \frac{1}{16}\epsilon^2 + \mathcal{O}(\epsilon^3) \right) t,$$

coincides exactly with the results found previously.

6. Amplitude equations

As we have found, the asymptotic solution of the vector system

$$\dot{\mathbf{x}} = \epsilon \mathbf{F}(\mathbf{x}, t, \epsilon), \tag{3.1}$$

on an appropriate $t = \mathcal{O}(\epsilon^{-1})$ interval can be obtained using averaging, multiple scales, or renormalization. Recall the regular perturbation (naive) expansion

$$\mathbf{x}_\epsilon(t) = \mathbf{x}_0(t) + \epsilon\mathbf{x}_1(t) + \epsilon^2\mathbf{x}_2(t) + \cdots. \tag{3.3}$$

The constant leading term $\mathbf{x}_0(t) = \mathbf{x}(0)$ is secular-free, but its first correction

$$\mathbf{x}_1(t) = \int_0^t \mathbf{F}(\mathbf{x}(0), s, 0) ds = \langle \mathbf{F}(\mathbf{x}(0), t, 0) \rangle \, t + \int_0^t \{\mathbf{F}(\mathbf{x}(0), s, 0)\} \, ds \tag{6.1}$$

is naturally split into its secular and bounded parts. The same is true for higher-order terms. By deleting all secular terms generated, we can immediately define the *bare* (or secular-free) expansion

$$\mathbf{x}_\epsilon(t) = \mathbf{x}(0) + \epsilon \int_0^t \{\mathbf{F}(\mathbf{x}(0), s, 0)\} \, ds + \mathcal{O}(\epsilon^2)$$

to any order. This sum, for a constant vector $\mathbf{x}(0)$, does not solve (3.1), however. When we instead replace $\mathbf{x}(0)$ by an appropriate slowly-varying amplitude $\mathbf{A}(\sigma)$, the approximation

$$\mathbf{x}_\epsilon(t) = \mathbf{A}(\sigma) + \epsilon \int_0^t \{\mathbf{F}(\mathbf{A}(\sigma), s, 0)\} \, ds + \cdots, \tag{6.2}$$

may be successful.

To derive the amplitude equations for the weakly nonlinear oscillator (1.1), we introduce polar coordinates to transform the oscillator equation into the standard form (3.1). We also introduce a strained fast time to eliminate the necessity of splitting the resulting amplitude equation for the phase when there is a nontrivial rest point. To this end, let

$$y = \rho \cos(\eta + \phi) \quad \text{and} \quad \dot{y} = -\rho \sin(\eta + \phi), \tag{6.3}$$

where

$$\eta = (1 + \epsilon \omega(\epsilon)) t,$$

for $\omega(\epsilon) = \omega_0 + \omega_1 \epsilon + \cdots$, is a strained fast time and ρ and ϕ are new time-varying polar coordinates. In contrast with multiple scales, as formulated in Section 2, we begin the straining of the fast time at $\mathcal{O}(\epsilon)$, rather than $\mathcal{O}(\epsilon^2)$. This choice is deliberate and ultimately allows us to choose the rest point of the phase equation to be zero, thus shifting the slow dynamics to the radius and significantly simplifying the resulting calculations. As with averaging, $\mathbf{x} = \binom{\rho}{\phi}$. By the chain rule, the transformed system then becomes

$$\frac{d\mathbf{x}}{dt} = \epsilon \mathbf{F}(\mathbf{x}, \eta, \epsilon), \tag{6.4}$$

with a prescribed initial vector, $\mathbf{x}(0)$ and the forcing

$$\mathbf{F}(\mathbf{x}, \eta, \epsilon) = \begin{pmatrix} \sin(\eta + \phi) \, f\left(\rho \cos(\eta + \phi), -\rho \sin(\eta + \phi)\right) \\ \frac{1}{\rho} \cos(\eta + \phi) f\left(\rho \cos(\eta + \phi), -\rho \sin(\eta + \phi)\right) - \omega(\epsilon) \end{pmatrix}.$$

This suggests that we seek a solution by introducing the *near-identity* transformation

$$\mathbf{x}(t) = \mathbf{A}(\sigma) + \epsilon \mathbf{U}(\mathbf{A}(\sigma), \eta, \epsilon) \tag{6.5}$$

for an amplitude $\mathbf{A}$ varying with the slow-time $\sigma = \epsilon t$. We require the scaled correction $\mathbf{U}$ to be analytic in $\mathbf{A}$, 2π-periodic in η and to possess an asymptotic series in ϵ. We also ask that $\mathbf{A}(0) = \mathbf{x}(0)$ so that $\mathbf{U}(\mathbf{A}(0), 0, \epsilon) = 0$. We shall also require an *amplitude equation*

$$\frac{d\mathbf{A}}{d\sigma} = \mathbf{H}(\mathbf{A}, \epsilon), \tag{6.6}$$

to be satisfied, for a yet to be determined autonomous function $\mathbf{H}$.

Recall that the leading term $\mathbf{x}_0$ in the two-time expansion only depends on σ, while its first correction $\mathbf{x}_1$ depends on σ through the leading amplitude A_0. Such an *ansatz* is also central to averaging. Moreover, it allows one to renormalize, bypassing both the identification of the secular terms of the naive expansion and their subsequent removal [49]. Such an *ansatz* is also made in Lighthill's method [50]. This *ansatz* is sometimes called the *amplitude equation* technique [51–53]. In the applied literature (including stability and bifurcation theory and pattern formation), such multiscale methods are typically used quite informally, but often effectively: for example Pedlosky obtains the amplitude equations for a resonant triad solution of Charney's barotropic equations of atmospheric dynamics [54] and Fujimura relates the multiple scale method and the Landau equation in the context of stability theory for fluid mechanics [55–57]. In these various contexts, the amplitude-equation approach is more general than averaging, though the methods coincide for the oscillator (1.1). Analogous transformations are central to the related method of *normal forms*, [58–60].

As with multiple scales, we will treat the two timescales σ and η independently. Substituting the *ansatz* (6.5) in the system (6.4) and applying the chain rule then implies the partial differential equation

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}\sigma} + \epsilon \frac{\partial \mathbf{U}}{\partial \mathbf{A}} \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}\sigma} + (1 + \epsilon\omega(\epsilon)) \frac{\partial \mathbf{U}}{\partial \eta} = \mathbf{F}(\mathbf{A} + \epsilon\mathbf{U}, \eta, \epsilon). \tag{6.7}$$

Substituting (6.6) in (6.7) and taking the average over $0 \le \eta \le 2\pi$ we obtain the integral equation

$$\mathbf{H}(\mathbf{A}, \epsilon) = \frac{1}{2\pi} \int_0^{2\pi} \left(\mathbf{F}(\mathbf{A} + \epsilon\mathbf{U}, s, \epsilon) - \epsilon \frac{\partial \mathbf{U}}{\partial \mathbf{A}} \mathbf{H}(\mathbf{A}, \epsilon) \right) \mathrm{d}s, \tag{6.8}$$

since $\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}\sigma}$ is independent of η, while boundedness of $\mathbf{U}$ requires $\frac{\partial \mathbf{U}}{\partial \eta}$ to be periodic with a zero average. Alternatively,

$$\mathbf{H}(\mathbf{A}, \epsilon) = \left(\mathbf{I} + \epsilon \left\langle \frac{\partial \mathbf{U}}{\partial \mathbf{A}} \right\rangle \right)^{-1} \langle \mathbf{F}(\mathbf{A} + \epsilon\mathbf{U}, \eta, \epsilon) \rangle,$$

which is analogous to (4.6) and (5.9). This identifies the resonant terms which, when integrated, cause secular behavior. By isolating such resonant terms, we can deal with them separately from the harmless, oscillatory remainder in (6.7).

Next, we solve (6.7) for $\frac{\partial \mathbf{U}}{\partial \eta}$, insert (6.6) and integrate from 0 to η to obtain a second integral equation,

$$(1 + \epsilon\omega(\epsilon)) \mathbf{U}(\mathbf{A}, \eta, \epsilon) = \int_0^\eta \left(\mathbf{F}(\mathbf{A} + \epsilon\mathbf{U}, s, \epsilon) - \left(\mathbf{I} + \epsilon \frac{\partial \mathbf{U}}{\partial \mathbf{A}} \right) \mathbf{H}(\mathbf{A}, \epsilon) \right) \mathrm{d}s, \tag{6.9}$$

which represents the bounded, oscillatory correction $\mathbf{U}$ in (6.5), since $\mathbf{H}$ is the average of the periodic function $\mathbf{F} - \epsilon \frac{\partial \mathbf{U}}{\partial \mathbf{A}} \mathbf{H}$. We shall iterate in this coupled pair of integral equations to define $\mathbf{H}$ and $\mathbf{U}$ asymptotically.

Assuming the series expansions

$$\mathbf{H}(\mathbf{A}, \epsilon) = \mathbf{H}_0(\mathbf{A}) + \epsilon\mathbf{H}_1(\mathbf{A}) + \epsilon^2\mathbf{H}_2(\mathbf{A}) + \cdots \tag{6.10}$$

and

$$\mathbf{U}(\mathbf{A}, \eta, \epsilon) = \mathbf{U}_0(\mathbf{A}, \eta) + \epsilon\mathbf{U}_1(\mathbf{A}, \eta) + \epsilon^2\mathbf{U}_2(\mathbf{A}, \eta) + \cdots, \tag{6.11}$$

we have that $\mathbf{F}$ has the resulting Taylor-series expansion

$$\mathbf{F}(\mathbf{A} + \epsilon\mathbf{U}, \eta, \epsilon) = \mathcal{F}_0(\mathbf{A}, \eta) + \epsilon\mathcal{F}_1(\mathbf{A}, \mathbf{U}_0, \eta) + \epsilon^2\mathcal{F}_2(\mathbf{A}, \mathbf{U}_0, \mathbf{U}_1, \eta) + \cdots. \tag{6.12}$$

Then substituting the power series (6.10–6.12) in the integral equations (6.8) and (6.9), averaging and collecting terms with like powers of ϵ, one finds to leading order that

$$\mathbf{H}_0\left(\mathbf{A}\right) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{F}_0\left(\mathbf{A}, s\right)\, \mathrm{d}s = \langle \mathcal{F}_0\left(\mathbf{A}, \eta\right)\rangle \tag{6.13}$$

and

$$\mathbf{U}_0\left(\mathbf{A}, \eta\right) = \int_0^{\eta}\left(\mathcal{F}_0\left(\mathbf{A}, s\right) - \mathbf{H}_0\left(\mathbf{A}\right)\right)\, \mathrm{d}s, \tag{6.14}$$

as found by the method of averaging (recall (4.12); see also [5, 33, 61]). At $\mathcal{O}(\epsilon^n)$, for each $n \geq 1$, we determine the convenient corresponding pair of recursion relations

$$\mathbf{H}_n\left(\mathbf{A}\right) = \left\langle \mathcal{F}_n\left(\mathbf{A}, \mathbf{U}_0, \ldots, \mathbf{U}_{n-1}, \eta\right)\right\rangle - \sum_{k=0}^{n-1}\left\langle \frac{\partial \mathbf{U}_k}{\partial \mathbf{A}}\right\rangle \mathbf{H}_{n-k-1}\left(\mathbf{A}\right) \tag{6.15}$$

and

$$\mathbf{U}_n\left(\mathbf{A}, \eta\right) = \int_0^{\eta}\Bigg[\mathcal{F}_n\left(\mathbf{A}, \mathbf{U}_0, \ldots, \mathbf{U}_{n-1}, s\right) - \mathbf{H}_n\left(\mathbf{A}\right)$$
$$- \sum_{k=0}^{n-1}\left(\frac{\partial \mathbf{U}_k}{\partial \mathbf{A}} \mathbf{H}_{n-k-1}\left(\mathbf{A}\right)\right)\Bigg]\, \mathrm{d}s - \sum_{k=0}^{n-1}\mathbf{U}_k\,\omega_{n-k-1}. \tag{6.16}$$

These relations asymptotically determine both the bounded, higher-order approximations to the solution (6.5) and the higher-order terms in the amplitude equation (6.6). To completely succeed, all that remains is to *solve* the nonlinear initial-value problem

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}\sigma} = \mathbf{H}\left(\mathbf{A}, \epsilon\right), \quad \mathbf{A}(0) = \mathbf{x}(0). \tag{6.17}$$

To asymptotically solve (6.17), we insert a regular perturbation series

$$\mathbf{A}(\sigma) = \mathbf{A}_0(\sigma) + \epsilon \mathbf{A}_1(\sigma) + \epsilon^2 \mathbf{A}_2(\sigma) + \cdots. \tag{6.18}$$

The limiting problem,

$$\frac{\mathrm{d}\mathbf{A}_0}{\mathrm{d}\sigma} = \mathbf{H}_0\left(\mathbf{A}_0\right), \quad \mathbf{A}_0(0) = \mathbf{x}(0), \tag{6.19}$$

is generally fully nonlinear. Wherever it can be solved, later terms $\mathbf{A}_k$, for $k \geq 1$, follow successively as solutions of a sequence of linearized problems

$$\frac{\mathrm{d}\mathbf{A}_k}{\mathrm{d}\sigma} = \frac{\partial \mathbf{H}_0}{\partial \mathbf{A}}\left(\mathbf{A}_0\right)\mathbf{A}_k + \alpha_k\left(\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{k-1}\right), \quad \mathbf{A}_k(0) = 0, \tag{6.20}$$

where $\alpha_k\left(\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{k-1}\right)$ is known from inserting the regular perturbation series (6.18) into (6.17) and Taylor expanding $\mathbf{H}$ for small ϵ. Using the fundamental matrix $\frac{\partial \mathbf{A}}{\partial \mathbf{x}(0)}$,

$$\mathbf{A}_k(\sigma) = \frac{\partial \mathbf{A}_0(\sigma)}{\partial \mathbf{x}(0)}\int_0^{\sigma}\left[\left(\frac{\partial \mathbf{A}_0(r)}{\partial \mathbf{x}(0)}\right)^{-1}\alpha_k\left(\mathbf{A}_0(r), \mathbf{A}_1(r), \ldots, \mathbf{A}_{k-1}(r)\right)\right]\, \mathrm{d}r \tag{6.21}$$

is defined whenever $\mathbf{A}_0(\sigma)$ is. Its behavior as $\sigma \to \infty$ follows according to various stability hypotheses on $\mathbf{A}_0$. By comparison with the related work of [39, 62, 63], we should expect the series for $\mathbf{H}$ and $\mathbf{U}$ to diverge as Gevrey series.

As a specific example, reconsider the van der Pol equation (2.25). Once we use the transformations (6.3) to rewrite the equation as a first-order system in standard form, we apply the recursion formulas to find the amplitude equations

$$
\frac{dR}{d\sigma} = \frac{1}{2}R\left(1 - \frac{1}{4}R^2\right) + \epsilon^2\left(-\frac{5}{128}R^3 + \frac{17}{768}R^5 - \frac{37}{12288}R^7\right)
$$
$$
+ \epsilon^4\left(\frac{81}{2048}R^3 - \frac{2899}{55296}R^5 + \frac{167197}{7077888}R^7 - \frac{105841}{23592960}R^9 + \frac{29177}{94371840}R^{11}\right) + \mathcal{O}(\epsilon^5),
$$

$$
\frac{d\Phi}{d\sigma} = -\omega_0 - \epsilon\left(\omega_1 + \frac{1}{8} - \frac{11}{32}R^2 + \frac{21}{256}R^4\right) - \epsilon^2\omega_2 - \epsilon^3\left(\omega_3 + \frac{1}{128} - \frac{59}{512}R^2\right)
$$
$$
+ \frac{5897}{36864}R^4 - \frac{18241}{294912}R^6 + \frac{5633}{786432}R^8\right) - \epsilon^4\omega_4 + \mathcal{O}(\epsilon^5).
$$

The leading terms agree exactly with the averaged equations derived earlier, except for the presence of the ω_j's in the phase. Here we have written the $\mathcal{O}(\epsilon^4)$ amplitude equations, but note that the recursion formulas and a computer-algebra system (such as MAPLE) make it quite straightforward to derive any desired number of terms. Regular perturbation series for $R(\sigma)$ and $\Phi(\sigma)$ imply the same $\mathcal{O}(\epsilon)$ solution derived in the averaging section, including identical phase shifts $\omega_0 = 0$ and $\omega_1 = -\frac{1}{16}$, chosen so that the $\mathcal{O}(\epsilon)$ term in the phase equation decays to zero as $\sigma \to \infty$. One may easily use these amplitude equations to calculate higher-order approximations to the solution of the van der Pol equation.

We may also use the amplitude equations to verify the limit-cycle solution found by the Poincaré-Lindstedt procedure. Recall that in this procedure one assumes both an initial value $y(0) = a_0 + \epsilon a_1 + \epsilon^2 a_2 + \cdots$ *on the limit cycle* and a strained coordinate $\eta = (1 + \epsilon\Omega_0 + \epsilon^2\Omega_1 + \cdots)t$ to account for the ϵ-dependent period of the phase plane orbit. One then chooses the constants a_k and Ω_k to eliminate secular terms from the regular perturbation expansion. The result is an approximation of the periodic solution that is uniformly valid on a $t = \mathcal{O}(\epsilon^{-1})$ interval to the specified order of error. To $\mathcal{O}(\epsilon^3)$, this periodic solution for the van der Pol equation is

$$
y(t) = 2\cos\eta + \epsilon\left(\frac{3}{4}\sin\eta - \frac{1}{4}\sin 3\eta\right) - \epsilon^2\left(\frac{1}{8}\cos\eta - \frac{3}{16}\cos 3\eta + \frac{5}{96}\cos 5\eta\right)
$$
$$
- \epsilon^3\left(\frac{7}{256}\sin\eta - \frac{21}{256}\sin 3\eta + \frac{35}{576}\sin 5\eta - \frac{7}{576}\sin 7\eta\right) + \mathcal{O}(\epsilon^4), \tag{6.22}
$$

where the initial value is

$$
y(0) = 2 + \frac{1}{96}\epsilon^2 + \mathcal{O}(\epsilon^4)
$$

(together with $\dot{y}(0) = 0$) and the strained coordinate is

$$
\eta = \left(1 - \frac{1}{16}\epsilon^2 + \frac{17}{3072}\epsilon^4 + \mathcal{O}(\epsilon^5)\right)t. \tag{6.23}
$$

The attracting nature of the limit cycle implies that any nontrivial solution will be captured as $t \to \infty$. Thus, we can expect the asymptotically stable rest point of the amplitude equation for the radius $R(\sigma)$ to coincide with the initial value determined with the Poincaré-Lindstedt procedure. Hence, we substitute

$$
R(\sigma) = R(\infty) \equiv R_0(\infty) + \epsilon^2 R_2(\infty) + \mathcal{O}(\epsilon^4)
$$

in the amplitude equations. Since the derivatives $\frac{\mathrm{d}R}{\mathrm{d}\sigma}$ and $\frac{\mathrm{d}\Phi}{\mathrm{d}\sigma}$ become zero at the rest point, we find $R(\infty) = 2 + \frac{1}{96}\epsilon^2 + \mathcal{O}(\epsilon^4)$, as expected, as well as the strained coordinate (6.23). Substituting this radius in the secular-free expansion, we determine exactly the limit-cycle approximation (6.22). Using the $\mathcal{O}(\epsilon^{10})$ amplitude equations instead, we may further determine

$$R(\infty) = 2 + \frac{1}{96}\epsilon^2 - \frac{1033}{552960}\epsilon^4 + \frac{1019689}{55738368000}\epsilon^6 + \frac{9835512276689}{157315969843200000}\epsilon^8$$
$$+ \frac{58533181813182818069}{732614178920988672000000000}\epsilon^{10} + \mathcal{O}(\epsilon^{11}) \tag{6.24}$$

and the improved strained coordinate

$$\eta = \left(1 - \frac{1}{16}\epsilon^2 + \frac{17}{3072}\epsilon^4 + \frac{35}{884736}\epsilon^6 - \frac{678899}{5096079360}\epsilon^8 + \frac{28160413}{2293235712000}\epsilon^{10} + \mathcal{O}(\epsilon^{12})\right) t. \tag{6.25}$$

Using (6.24) and (6.25) in the secular-free expansion calculated with the $\mathcal{O}(\epsilon^{10})$ amplitude equations, gives a limit-cycle solution with an error of $\mathcal{O}(\epsilon^{11})$ on the $\mathcal{O}(\epsilon^{-1})$ timescale. As independent confirmation, we note that the frequency of the van der Pol oscillator was calculated to very high orders in [64].

When we have a periodic solution, we can use the amplitude equations to provide a solution on a time interval much longer than the usual $t = \mathcal{O}(\epsilon^{-1})$. Specifically, the rest points of the amplitude equations then provide the radius of the limit cycle and frequency of the oscillator. Suppose we have an $\mathcal{O}(\epsilon^n)$ approximation

$$y(t; \epsilon) = y_0(\eta, \sigma) + \epsilon y_1(\eta, \sigma) + \cdots + \epsilon^n y_n(\eta, \sigma) + \mathcal{O}(\epsilon^{n+1}),$$

with a purported $\mathcal{O}(\epsilon^{n+1})$ error, assuming η is completely determined. In practice, however, we only know the frequency to $\mathcal{O}(\epsilon^k)$, for some $k > n$. We may then approximate

$$\eta = \left(1 + \epsilon \omega_0 + \epsilon^2 \omega_1 + \cdots + \epsilon^k \omega_{k-1} + \mathcal{O}(\epsilon^{k+1})\right) t \equiv \eta_\epsilon^k + \mathcal{O}(\epsilon^{k+1}t),$$

so that

$$y(t; \epsilon) = y_0(\eta_\epsilon^k, \sigma) + \epsilon y_1(\eta_\epsilon^k, \sigma) + \cdots + \epsilon^n y_n(\eta_\epsilon^k, \sigma) + \mathcal{O}(\epsilon^{n+1}) + \mathcal{O}(\epsilon^{k+1}t), \tag{6.26}$$

since the y_j's and their derivatives are all uniformly bounded when $y(t; \epsilon)$ is periodic. For $k = n + 1$, we have the usual $\mathcal{O}(\epsilon^{-1})$ time interval of validity; *i.e.*, we must calculate the strained coordinate to one higher order in ϵ than our proposed approximation in order to maintain the same level of error. For the van der Pol results derived above, we may have an error of $\mathcal{O}(\epsilon^{11})$ on the $\mathcal{O}(\epsilon^{-1})$ timescale, as previously noted, or we could use the $\mathcal{O}(\epsilon^3)$ approximation, (6.22), and the $\mathcal{O}(\epsilon^{11})$ strained coordinate, (6.25), to achieve an $\mathcal{O}(\epsilon^4)$ error on the $\mathcal{O}(\epsilon^{-8})$ timescale. This "trade-off" principle for (6.26) was noted by Murdock in [4], but was likely known informally long before. Rubenfeld, [65], developed a method with a similar, but more limited, trade-off principle.

Using a regular perturbation series in the amplitude equations does not always give the correct result. This procedure is vulnerable when the weakly nonlinear forcing of (1.1) also depends on ϵ, *i.e.*, when the forcing is $f(y, \dot{y}, \epsilon)$. Morrison, [21], introduced the problem

$$\ddot{y} + y + \epsilon \dot{y}^3 + 3\epsilon^2 \dot{y} = 0, \tag{6.27}$$

to show how the method of multiple scales is similarly vulnerable. Fortunately, the amplitude equation method allows us to overcome these difficulties by simply modifying the procedure used to solve the amplitude equations. If a regular perturbation series is unsatisfactory,

e.g. if $\mathbf{A}_j(\sigma)$ blows up for some $j \geq 1$, one may instead attempt to simply rescale the amplitude equations (*cf.* [18, Section 4.2.4], where an entirely new multiple scale expansion must be derived). Once a consistent rescaling has been found, the problem for the rescaled amplitude equations may then be solved using a regular perturbation series, as before.

For (6.27), we obtain the amplitude equations

$$\frac{\mathrm{d}R}{\mathrm{d}\sigma} = -\frac{3}{8}R^3 - \epsilon\frac{3}{2}R - \epsilon^2\frac{45}{4096}R^7 - \epsilon^3\frac{9}{256}R^5 + \epsilon^4\left(\frac{81}{128}R^3 + \frac{267651}{10485760}R^{11}\right) + \mathcal{O}(\epsilon^5),$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\sigma} = -\omega_0 - \epsilon\left(\omega_1 + \frac{45}{256}R^4\right) - \epsilon^2\left(\omega_2 + \frac{27}{32}R^2\right) - \epsilon^3\left(\omega_3 + \frac{9}{8} - \frac{6021}{262144}R^8\right)$$
$$- \epsilon^4\left(\omega_4 - \frac{3735}{32768}R^6\right) + \mathcal{O}(\epsilon^5).$$

A regular perturbation solution of these equations gives the leading amplitude

$$R_0(\sigma) = \frac{2}{\sqrt{3\sigma + 4}},$$

in agreement with multiple scales. Unfortunately, this algebraically decaying amplitude underestimates the decay to the rest point. Worse, the $\mathcal{O}(\epsilon)$ correction $R_1(\sigma)$ blows up like $\sqrt{\sigma}$ as $\sigma \to \infty$, as do the multiple scales results. Figure 1 shows the multiple scale approximation plotted together with a numerical solution of (6.27). The deficient decay rate is evident. We may attempt to salvage our amplitude equations by introducing a rescaling transformation

$$Z = \epsilon^\alpha R, \quad \kappa = \epsilon^\beta \sigma,$$

in an attempt to find a more appropriate timescale over which the decay occurs. Using the transformations in the amplitude equations and applying dominant balance arguments implies that the only new consistent balance is $\alpha = -\frac{1}{2}$ and $\beta = 1$. Defining $\Psi(\kappa) = \Phi\left(\frac{\kappa}{\epsilon}\right)$, we find the rescaled amplitude equations

$$\frac{\mathrm{d}Z}{\mathrm{d}\kappa} = -\frac{3}{8}Z^3 - \frac{3}{2}Z + \epsilon^4\left(\frac{81}{128}Z^3 - \frac{9}{256}Z^5 - \frac{45}{4096}Z^7\right) + \mathcal{O}(\epsilon^8),$$
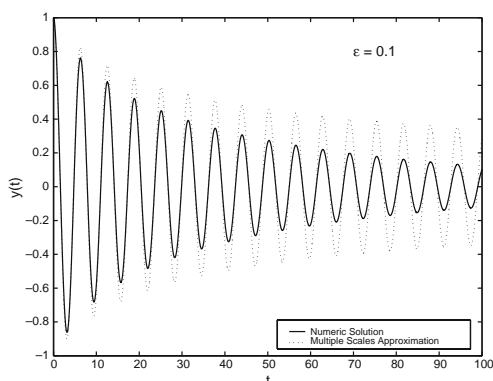

Figure 1. The numeric solution of Morrison's equation and the multiple-scales approximation. Note how the multiple-scales solution fails to decay quickly enough to the rest point at zero.
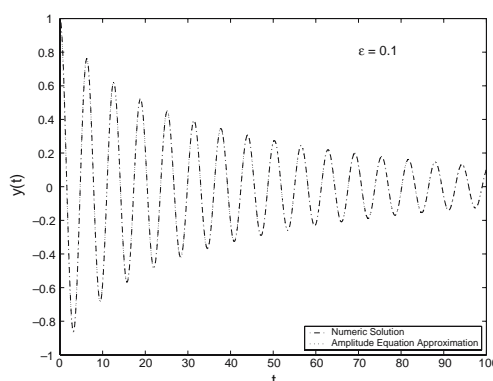

Figure 2. The numeric solution of Morrison's equation and the rescaled amplitude equation approximation are indistinguishable from each other.

$$\frac{d\Psi}{d\kappa} = -\frac{\omega_0}{\epsilon} - \omega_1 - \epsilon\omega_2 + \epsilon^2\left(\omega_3 + \frac{9}{8} + \frac{27}{32}Z^2 + \frac{45}{256}Z^4\right) - \epsilon^3\omega_4 - \epsilon^4\omega_5 + \mathcal{O}(\epsilon^5).$$

We now solve the rescaled amplitude equations using the regular perturbation series

$$Z(\kappa) = Z_0(\kappa) + \epsilon^4 Z_1(\kappa) + \mathcal{O}(\epsilon^8).$$

The leading-order problem is the Bernoulli equation

$$\frac{dZ_0}{d\kappa} = -\frac{3}{8}Z_0^3 - \frac{3}{2}Z_0, \quad Z_0(0) = \frac{1}{\sqrt{\epsilon}},$$

whose solution

$$Z_0(\kappa) = \frac{2}{\sqrt{(1+4\epsilon)\,e^{3\kappa} - 1}},$$

decays exponentially to the trivial rest point as $\kappa \to \infty$. Further, to eliminate linearly growing terms in the phase, we must set $\omega_0 = \omega_1 = \omega_2 = \omega_4 = \omega_5 = 0$ and $\omega_3 = -\frac{9}{8}$. An integration then implies that

$$\Psi(\kappa) = \epsilon^2\left(\frac{15}{64}\left(Z_0^2 - 1\right) + \frac{3}{16}\log\left(\frac{1}{5}\left(Z_0^2 + 4\right)\right)\right) + \mathcal{O}\left(\epsilon^6\kappa\right).$$

Thus we may write the final approximation as

$$y(t) = \sqrt{\epsilon}Z_0\left(\cos\eta + \epsilon^2\left[\left(\frac{33}{64}Z_0^2 + \frac{81}{64} + \frac{3}{16}\log\left(\frac{1}{5}\left(Z_0^2 + 4\right)\right)\right)\sin\eta + \frac{1}{32}Z_0^3\sin 3\eta\right] + \mathcal{O}(\epsilon^4)\right).$$

Note that the expansion is not singular at $\kappa = 0$ as $\epsilon \to 0$ since $\sqrt{\epsilon}Z_0$ is bounded. This modified amplitude equation solution is plotted with the numerical solution in Figure 2. Owing to the asymptotic stability of the rest point, the approximation is valid for all $t \geq 0$. Analogous results follow for the van der Pol-Duffing equation

$$\ddot{y} + y + \epsilon y^3 + \epsilon^2(y^2 - 1)\dot{y} = 0, \tag{6.28}$$

a stochastic version of which is considered in [66]. The interested reader may find a more complete discussion of rescaling for systems of ordinary differential equations in [67].

We note that our approach, based on the elimination of resonant terms, is analogous to the use of various initialization methods to account for boundary layers [68–70]. Thus, we can expect the amplitude equation method to encompass such problems as well. Many more fascinating oscillator-type examples may be found in the texts [26, 71–72].

7. Concluding remarks

In this paper, we have studied a variety of techniques for approximating solutions to the weakly nonlinear oscillator (1.1). Each was demonstrated on a common set of nontrivial examples and the different approximation methods were shown to give identical results. Though each approach considered has advantages, we believe the amplitude-equation method ultimately provides the greatest flexibility.

A major benefit of the amplitude equation technique is the ease with which it can be automated by a computer-algebra system. The recursion formulas we derived make such automation exceptionally easy. Once a working code has been produced and tested (as the authors

have done with MAPLE), the computational work to derive the amplitude equations for any specific weakly nonlinear oscillator is essentially trivial except for the computer processor time involved.

A second major benefit of the amplitude-equation technique is that the ϵ-dependent frequency may be quite easily derived directly from the amplitude equations. In the case of periodic solutions, one may use this frequency to define an asymptotic approximation that is valid for much longer time intervals than is customary with either multiple scales or averaging. Although one derives a strained coordinate using multiple scales, new terms of the frequency expansion are found one at a time, so the convenience of knowing the frequency to high orders is lost. Likewise, it is not customary to obtain a strained coordinate with averaging, so those results are more limited. Our results clearly represent an advance over these two methods.

An important application of these methods is to provide or validate related algorithms to obtain numerical solutions to initial-value problems over long time intervals. The state of the art for these efforts is described in [73].

Acknowledgements

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